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Automorphism groups of $\mathbb{P}^1$-bundles over a non-uniruled base

T. Bandman, Yu. G. Zarhin

In this survey we discuss holomorphic $\mathbb{P}^1$-bundles $p\colon X \to Y$ over a non-uniruled complex compact Kähler manifold Y, paying a special attention to the case when Y is a complex torus. We consider the groups $\mathrm{Aut}(X)$ and $\mathrm{Bim}(X)$ of its biholomorphic and bimeromorphic automorphisms, respectively, and discuss when these groups are bounded, Jordan, strongly Jordan, or very Jordan.

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1. Introduction

In this survey we consider the groups $\mathrm{Aut}(X)$ and $\mathrm{Bim}(X)$ of all biregular and bimeromorphic self-maps, respectively, for a compact complex connected Kähler manifold X. If X is projective, then $\mathrm{Bim}(X) = \mathrm{Bir}(X)$ is the group of all birational transformations of X (see [74]). The manifolds we are going to deal with are of special type: X has to be a $\mathbb{P}^1$-bundle over a non-uniruled compact complex connected manifold Y.

In general, the groups $\mathrm{Bim}(X)$ may be very huge and non-algebraic (for example, the Cremona group Cr_n of birational transformation of the n-dimensional projective space). Thus one is tempted to study the properties of a group via its finite and/or abelian subgroups. Namely, we are interested in the following properties of groups.

Definition 1.1. (a) A group G is called *bounded* if the orders of its finite subgroups are bounded by a universal constant that depends only on G (see [58; Definition 2.9]).

(b) A group G is called *Jordan* if there is a positive integer J such that every finite subgroup B of G contains an abelian subgroup A that is normal in B and such that the index $[B : A] \leqslant J$. The smallest such J is called the *Jordan constant* of G and is denoted by J_G (see [76; Question 6.1], [58; Definition 2.1], [59]).

(c) A Jordan group G is called *strongly Jordan* [62], [5] if there is a positive integer m such that every finite subgroup of G is generated by at most m elements.

(d) A group G is *very Jordan* [7] if there exist a commutative normal subgroup G_0 of G and a bounded group F that sit in a short exact sequence

$$1 \to G_0 \to G \to F \to 1. \tag{1}$$

In what follows by a *Jordan property* we mean any one of the properties described in Definition 1.1. The study of these properties has been inspired by the following fundamental results.

Theorem 1.2 (M.-E.-C. Jordan (1878), [32], [77; Theorem 9.9]). *Let $\mathbb{C}$ be the field of complex numbers. Then $\mathrm{GL}(n, \mathbb{C})$ is strongly Jordan.*

Theorem 1.3 (J.-P. Serre (2009), [76; Theorem 5.3]). $\mathrm{Cr}_2 = \mathrm{Bir}(\mathbb{P}^2)$ *is Jordan*, $J_{\mathrm{Cr}_2} \leqslant 2^{10}\, 3^4\, 5^2\, 7$.

It was V. L. Popov who asked in [58] a question whether for an algebraic variety X the groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ are Jordan. This question originated an intensive and fruitful activity. It was proved that there are vast classes of manifolds (varieties) with Jordan groups $\mathrm{Aut}(X)$, $\mathrm{Bim}(X)$, and $\mathrm{Bir}(X)$; see § 4. In particular, the Cremona group $\mathrm{Cr}_n = \mathrm{Bir}(\mathbb{P}^n)$ appeared to be Jordan for all n ([62] and [10]; this is the positive answer to a question formulated by J.-P. Serre). In § 4 we give a glimpse on the richness of known facts about the Jordan properties of $\mathrm{Aut}(X)$, $\mathrm{Bim}(X)$, or $\mathrm{Bir}(X)$ for various types of varieties X. We do not pretend to give a complete picture. Our aim is to demonstrate that the "worst" manifolds from this point of view are uniruled but not rationally connected ones. For example, the group $\mathrm{Bim}(X)$ is not Jordan if X is bimeromorphic to a product of a complex torus of positive algebraic dimension and the projective space $\mathbb{P}^N$, $N > 0$ (see [85], [87]).

In this survey we concentrate on the manifolds of this kind. Namely, our main object of consideration are $\mathbb{P}^1$-bundles over non-uniruled manifolds, that is, triples (X, p, Y) such that

- X, Y are compact complex connected Kähler manifolds;
- $p\colon X \to Y$ is a holomorphic map from X onto Y;
- Y is not uniruled;
- for every point $y \in Y$ the fibre $p^*(y)$ is isomorphic to $\mathbb{P}^1$; in particular, it is irreducible and reduced.

We say that such a triple (X, p, Y) has an *almost section* D if an irreducible analytic subset $D \subset X$, $\mathrm{codim}(D) = 1$, meets a general fibre of p at precisely one point (see Definition 6.5). We say that such a triple (X, p, Y) (or X, or the morphism p) is *scarce* if X does *not* admit three distinct almost sections A_1, A_2, A_3 such that $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3$ (see Definition 11.5). We say that a connected compact complex manifold Y is *poor* (Definition 13.1) if it contains neither rational curves nor analytic subsets of codimension 1.

The facts that we know about Jordan properties of $\mathbb{P}^1$-bundles (X, p, Y) over non-uniruled Kähler manifolds are summarized below.

Summary. 1) $\mathrm{Aut}(X)$ is always Jordan ([34], for surfaces see also [86]) and even strongly Jordan (see Remark 4.1).

2) If morphism p is scarce, then $\mathrm{Aut}(X)$ is very Jordan (Theorem 12.1 of this paper).

3) If Y is a torus and X is not a projectivization of a decomposable vector bundle of rank 2 on Y, then the group $\mathrm{Aut}(X)$ is strongly Jordan [78].

4) If X and Y are projective, and X is not birational to $Y \times \mathbb{P}^1$, then $\mathrm{Bir}(X)$ is strongly Jordan [5].

5) If Y is a poor manifold (see Definition 13.1), then $\mathrm{Bim}(X)$ coincides with $\mathrm{Aut}(X)$ and is very Jordan [7].

6) If Y is a complex torus and there is no almost section of p, then $\mathrm{Bim}(X)$ is Jordan [78]. In particular, if X is not the projectivization of a rank 2 vector bundle on Y, then the group $\mathrm{Bim}(X)$ is strongly Jordan.

7) If Y is a complex torus of positive algebraic dimension and X is bimeromorphic (birational, if Y is projective) to a direct product $Y \times \mathbb{P}^1$, then the group $\mathrm{Bim}(X)$ (respectively, $\mathrm{Bir}(X)$) is not Jordan [85], [87].

8) If Y is a complex torus of positive algebraic dimension, Y_a is its algebraic reduction, $\mathcal{L}$ is the lift to Y of a holomorphic line bundle on Y_a, and X is the projectivization of the rank 2 vector bundle $\mathcal{L} \oplus \mathbf{1}$, then $\mathrm{Bim}(X)$ is not Jordan [87].

9) *Open question.* Assume that Y is a complex torus of positive algebraic dimension and X has no representation as in previous item. Is $\mathrm{Bim}(X)$ Jordan?

Our goal is to give a review of the methods used to prove these facts. The results unpublished previously are provided with full proofs.

All manifolds are compact complex, and connected unless otherwise stated. All algebraic varieties are complex, projective, irreducible, and reduced. $\mathbb{P}^n$ and $\mathbb{C}^n$ are complex projective and affine spaces, respectively; $\mathbb{P}^n_k$ and $\mathbb{C}^n_k$ are projective and affine spaces, respectively, over an algebraically closed field k.

The structure of the survey is as follows. In § 2 we provide facts and examples concerning bounded, Jordan, and very Jordan groups. In § 3 we enumerate the assumptions and notation and recall the notions related to manifolds and their maps. In § 4 we give examples of the known facts about the Jordan properties of $\mathrm{Aut}(X)$, $\mathrm{Bim}(X)$, and $\mathrm{Bir}(X)$ for various types of manifolds X. Our aim is to demonstrate a special role of $\mathbb{P}^1$-bundles over a non-uniruled base in this field. In § 5 we provide some generalities on maps of $\mathbb{P}^1$-bundles. In § 6 we deal with the group $\mathrm{Bim}(X)$ of a non-trivial rational bundle (in particular, projective conic bundle). In Chapter 3 we deal with certain $\mathbb{P}^1$-bundles over complex tori. We present a unified approach to proving results of [85] and [87]. It is based on symplectic algebra, which offers highly useful tools for studying line bundles over tori, and is inspired by the work of D. Mumford [46]. In Chapter 4 we consider $\mathbb{P}^1$-bundles (X, p, Y) with scarce sets of sections over a non-uniruled Kähler base. That chapter presents a generalization and a modification of [7]. First, in § 11, for a $\mathbb{P}^1$-bundle (X, p, Y) we consider the group $\mathrm{Aut}(X)_p$ of those automorphisms of X that leave every fibre of p fixed. In three subsections we describe three different types of such automorphisms. In § 12, under the assumption that Y is Kähler and not uniruled and p is scarce, we prove that the neutral component $\mathrm{Aut}_0(X)$ of the complex Lie group $\mathrm{Aut}(X)$ is commutative, hence $\mathrm{Aut}(X)$ is very Jordan. In § 13 we prove that if Y is poor, then p is scarce and $\mathrm{Aut}(X)$ is very Jordan.

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Chapter 1. Preliminaries

In this chapter we provide some backgrounds: the properties of Jordan groups, the notation, assumptions, and definitions.

2. Jordan properties of groups

In this section we recall the general facts about the Jordan properties of groups. The following properties are easy consequences of Definition 1.1.

1) Every finite group is bounded, Jordan, and very Jordan.

2) Every commutative group is Jordan and very Jordan.

3) Every finitely generated commutative group is bounded. Indeed, such a group is isomorphic to a finite direct sum with every summand isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$, where n is a positive integer.

4) A subgroup of a Jordan group is Jordan. A subgroup of a very Jordan group is very Jordan.

5) "Bounded" implies "very Jordan", "very Jordan" implies "Jordan".

6) "Bounded" implies "strongly Jordan". On the other hand, "very Jordan" does not imply "strongly Jordan". For example, a direct sum of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$ is commutative but has finite subgroups with any given minimal number of generators.

Example 2.1. The group $\mathrm{GL}(n, \mathbb{Z})$ is bounded. This is a consequence of the following theorem of Minkowski [77; § 9.1].

Theorem 2.2 (Minkowski, 1887). *If an element $a \in \mathrm{GL}(n, \mathbb{Z})$ is periodic, and $a = \mathbf{1} \bmod m$ for $m \geqslant 3$, then $a = 1$.*

It follows that every finite subgroup $H \subset \mathrm{GL}(n, \mathbb{Z})$ embeds into $\mathrm{GL}(n, \mathbb{Z}/3\mathbb{Z})$ (there are much more precise bounds: see [75; Theorem 1.1]). Since every finite subgroup of $\mathrm{GL}(n, \mathbb{Q})$ is conjugate to a subgroup of $\mathrm{GL}(n, \mathbb{Z})$ (see [75; Lecture 1]), the group $\mathrm{GL}(n, \mathbb{Q})$ is also bounded.

Example 2.3. The multiplicative group $\mathbb{C}^*$ of $\mathbb{C}$ is commutative, very Jordan but not bounded. The same is valid for the group of translations of a complex torus of positive dimension.

Example 2.4. From Theorem 1.2 it follows that the group $\mathrm{GL}(n, k)$ is strongly Jordan for every field k of characteristic zero. Moreover, every linear algebraic group over k is strongly Jordan. On the other hand $\mathrm{GL}(n, k)$ is obviously not very Jordan if $n \geqslant 2$.

The following precise values of Jordan constants for groups $\mathrm{GL}(n, \mathbb{C})$ were found by M. J. Collins.

Theorem 2.5 [18; Theorems **A** and **B**]. *For the Jordan constants of groups $\mathrm{GL}(n, \mathbb{C})$ the following relations hold*:

(i) $J_{\mathrm{GL}(n,\mathbb{C})} = (n + 1)!$ *if* $n \geqslant 71$ *or* $n = 63, 65, 67, 69$;

(ii) $J_{\mathrm{GL}(n,\mathbb{C})} = 60^r \cdot r!$ *if* $20 \leqslant n \leqslant 62$ *or* $n = 64, 66, 68, 70$ *where* $r = [n/2]$.

The information on the values of Jordan constants for the groups $\mathrm{GL}(n, \mathbb{C})$, $n < 20$, is given in extensive tables provided in the same paper.

Example 2.6. We use below analogues of the Heisenberg groups that were used by D. Mumford [46]. Let

- $\mathbf{K}$ be a finite commutative group of order $N > 1$;
- $\mu_N \subset \mathbb{C}^*$ be the multiplicative group of Nth roots of unity;
- $\widehat{\mathbf{K}} = \mathrm{Hom}(\mathbf{K}, \mu_N)$ (the dual of $\mathbf{K}$).

Mumford's theta group $\mathfrak{G}_{\mathbf{K}}$ for $\mathbf{K}$ is the group of matrices of type

$$\begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha \in \widehat{\mathbf{K}}$, $\gamma \in \mathbb{C}^*$, and $\beta \in \mathbf{K}$. The product of $\alpha \in \widehat{\mathbf{K}}$ and $\beta \in \mathbf{K}$ is defined by a certain natural non-degenerate alternating bilinear form $e_{\mathbf{K}}$ on $\mathbf{H}_{\mathbf{K}} = \mathbf{K} \times \widehat{\mathbf{K}}$

with values in $\mathbb{C}^*$ [85; p. 302]. This group can be included in a short exact sequence

$$1 \to \mathbb{C}^* \to \mathfrak{G}_{\mathbf{K}} \to \mathbf{H}_{\mathbf{K}} \to 1$$

where the image of $\mathbb{C}^*$ is the center of $\mathfrak{G}_{\mathbf{K}}$.

Properties of $\mathfrak{G}_{\mathbf{K}}$ [85; p. 302] imply that it is a *theta group* attached to the *non-degenerate symplectic pair* $(\mathbf{H}_{\mathbf{K}}, e_{\mathbf{K}})$ in the sense of Chapter 3 below. By Theorem 7.17 below, $\mathfrak{G}_{\mathbf{K}}$ is Jordan and

$$J_{\mathfrak{G}_{\mathbf{K}}} = \sqrt{\#(\mathbf{H}_{\mathbf{K}})} = N = \#(\mathbf{K}).$$

In particular, if $K = \mathbb{Z}/N\mathbb{Z}$, then $J_{\mathfrak{G}_{\mathbb{Z}/N\mathbb{Z}}} = N$.

Example 2.7. The example of a non-Jordan group is given by $\mathrm{SL}(2, \overline{\mathbb{F}}_p)$, where $\overline{\mathbb{F}}_p$ is the algebraic closure of a prime finite field $\mathbb{F}_p$ with p elements.

Indeed, if $q = p^n \geqslant 4$, then $\mathrm{SL}(2, \mathbb{F}_q) \subset \mathrm{SL}(2, \overline{\mathbb{F}}_p)$ (here $\mathbb{F}_q$ is the q-element field). The group $\mathrm{SL}(2, \mathbb{F}_q)$ is non-commutative, finite, and has order $(q^2-1)q$. Every normal subgroup $C \subsetneq \mathrm{SL}(2, \mathbb{F}_q)$ consists of one or two scalars, thus the indices

$$[\mathrm{SL}(2, \mathbb{F}_q) : C] = \frac{1}{2}(q^2-1)q \quad \text{or} \quad (q^2-1)q$$

are unbounded as n tends to infinity. Hence $\mathrm{SL}(2, \overline{\mathbb{F}}_p)$ is not Jordan.

Remark 2.8. An analogue of Jordan's theorem holds for matrix groups over fields k of prime characteristic p if one considers only finite subgroups, whose order is prime to p. On the other hand there are generalizations of Jordan's theorem (Brauer–Feit [14], Larsen–Pink [38]) that deal with arbitrary finite subgroups and take the orders of their Sylow p-subgroups into account. These results have led to the following definition [30; Definition 1.6] (it will be used in Remark 4.3(iv) below).

A group G is called *p-Jordan* if there exist positive integers J and e such that every finite subgroup B of G contains an abelian p'-subgroup A that is normal in B and such that the index $[B : A] \leqslant |B_p|^e J$. Here $|B_p|$ is the order of a Sylow p-subgroup of B.

Remark 2.9. Let G be a group. Assume that it can be included into the following exact sequence of groups

$$0 \to H \to G \to F \to 0.$$

Then the following hold:

(i) if F is bounded and H is bounded, then G is bounded;
(ii) if H is very Jordan and F is bounded, then G is very Jordan;
(iii) if F is bounded, then G is Jordan if and only if H is Jordan [58; Lemma 2.11];
(iv) if H is bounded and F is strongly Jordan, then G is Jordan [62; Lemma 2.8];
(v) G being Jordan *does not* imply that F is Jordan;
(vi) F and H being Jordan *does not* imply that G is Jordan.

We will need the following modification of Lemma 5.3 in [7].

LEMMA 2.10. *Consider a short exact sequence of connected complex Lie groups:*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} D \to 0.$$

Here i is a closed holomorphic embedding and j is surjective and holomorphic. Assume that D is a complex torus and A is isomorphic as a complex Lie group either to $(\mathbb{C}^+)^n$ or to $\mathbb{C}^*$. Then B is commutative.

PROOF. The proof of this lemma coincides verbatim with the proof of Lemma 5.3 in [7] where the case $n = 1$ is treated.

Step 1. First, let us prove that A is a central subgroup in B. Take any element $b \in B$. Define a holomorphic map $\phi_b \colon A \to A$, $\phi_b(a) = bab^{-1} \in A$ for an element $a \in A$. Since it depends holomorphically on b, we have a holomorphic map $\xi \colon B \to \mathrm{Aut}(A)$, $b \to \phi_b$.

Since A is commutative, for every $c \in A$ we have $\phi_{bc} = \phi_b$. Thus there is a well-defined map ψ fitting into the following commutative diagram:

$$\begin{tikzcd} & B \arrow[dl, "j"'] \arrow[dr, "\xi"] & \\ D \arrow[rr, "\psi"] & & \mathrm{Aut}(A) \end{tikzcd}$$

The map $\psi = \xi \circ j^{-1}$ is defined at every point of D. It is holomorphic (see, for example, [55; § 3]).

Since D is a complex torus, and $\mathrm{Aut}(A)$ is either $\mathrm{GL}(n, \mathbb{C})$ (if $A = (\mathbb{C}^+)^n$) or consists of two elements, id and $a \mapsto a^{-1}$ (if $A = \mathbb{C}^*$), we have $\psi(D) = \{\mathrm{id}\}$. It follows that A is a central subgroup of B.

Step 2. Now let us show that B is commutative. Consider a holomorphic map $\mathrm{com} \colon B \times B \to A$ defined by $\mathrm{com}(x, y) = xyx^{-1}y^{-1}$. Since A is a central subgroup of B, similarly to Step 1 we obtain a holomorphic map $D \times D \to A$. It has to be constant since D is a torus and A is either $(\mathbb{C}^+)^n$ or $\mathbb{C}^*$.

3. Complex manifolds

This section contains preliminaries, the notation, and the assumptions that will be used further on.

By a (*projective*) *variety* we mean an algebraic variety that is a Zariski closed subset of a projective space $\mathbb{P}^n$.

Let $U \subset \mathbb{C}^n$ be an open subset. An analytic subset of U is a closed subset $X \subset U$ such that for any $x \in X$ there exist an open neighbourhood $x \in V \subset U$ and holomorphic functions $f_1, \dots, f_k \colon V \to \mathbb{C}$ such that $X \cap V = \{f_1 = 0, \dots, f_k = 0\}$ [31; Definition 1.1.23].

A complex space consists of a Hausdorff topological space X and a sheaf of rings $\mathcal{O}_X$ such that $(X, \mathcal{O}_X)$ is locally isomorphic to an analytic subset $Z \subset U \subset \mathbb{C}^n$ endowed with the sheaf $\mathcal{O}_U/\mathcal{I}$, where $\mathcal{I}$ is a sheaf of holomorphic functions such that $Z = Z(\mathcal{I})$ (see [31; Definition 6.2.8]). By Chow's theorem any closed analytic subset of complex projective space is a projective variety [28; Chap. V, Section D, Theorem 7], [74; Proposition 13].

A complex manifold is a complex space which is locally modelled on $Z = U \subset \mathbb{C}^n$ and $\mathcal{I} = \{0\}$ [31; Example 6.2.9]. In particular, it is smooth.

We will use the following notation and assumptions.

Notation and assumptions.

(NA.1) $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ stand for the ring of integers and fields of rational, real, and complex numbers, respectively.

(NA.2) In what follows, the ground field is $\mathbb{C}$ unless indicated otherwise.

(NA.3) $\mathrm{Aut}(X)$ stands for the group of all biholomorphic (or biregular, if X is projective) automorphisms of a complex manifold X. The group $\mathrm{Aut}(X)$ of any connected complex compact manifold X carries the natural structure of a complex (not necessarily connected) Lie group such that the action map $\mathrm{Aut}(X) \times X \to X$ is holomorphic (the Bochner–Montgomery theorem [13]).

(NA.4) $\mathrm{Aut}_0(X)$ stands for the connected identity component of $\mathrm{Aut}(X)$ (as a complex Lie group).

(NA.5) If $p\colon X \to Y$ is a morphism of complex manifolds, then $\mathrm{Aut}(X)_p$ is the subgroup of all $f \in \mathrm{Aut}(X)$ such that $p \circ f = p$.

(NA.6) For $f \in \mathrm{Aut}(X)$ we denote by $\mathrm{Fix}(f)$ the set of all fixed points of f.

(NA.7) $\cong$ stands for “isomorphic groups” (or isomorphic complex Lie groups if so are the groups involved), and also for isomorphic line bundles; $\sim$ stands for biholomorphically isomorphic complex manifolds; $\approx$ stands for bimeromorphic or birational complex manifolds.

(NA.8) id stands for the identity automorphism, $\mathbf{I}$ stands for an identity matrix.

(NA.9) We say that a subset U of a complex manifold X is analytically Zariski open if $U = X \setminus Z$, where Z is an analytic subspace of X.

(NA.10) $\mathbb{P}^n_{(x_0:\cdots:x_n)}$ stands for $\mathbb{P}^n$ with homogeneous coordinates $(x_0 : \cdots : x_n)$.

(NA.11) $\mathbb{C}_z$, $\overline{\mathbb{C}}_z \sim \mathbb{P}^1$ is the complex line (extended complex line, respectively) with coordinate z.

(NA.12) $\mathbb{C}^+$ and $\mathbb{C}^*$ stand for the complex Lie groups $\mathbb{C}$ and $\mathbb{C}^*$ with additive and multiplicative group structure, respectively.

(NA.13) $\dim(X)$ and $\dim_a(X)$ are the complex and algebraic dimensions of a compact complex manifold X, respectively.

(NA.14) We let $\mathbf{pr}$ denote the natural projection $Y \times \mathbb{P}^1 \to Y$.

(NA.15) For an element $\psi \in \mathrm{PSL}(2,\mathbb{C})$ we denote by $\mathrm{TD}(\psi)$ the number

$$\mathrm{TD}(\psi) := \frac{\mathrm{tr}(F)^2}{\det(F)},$$

where $F \in \mathrm{GL}(2,\mathbb{C})$ is any representative of ψ, and tr and det stand for the trace and determinant, respectively.

(NA.16) A rational curve in X is the image of a non-constant holomorphic map $\mathbb{P}^1 \to X$.

(NA.17) $\mathbf{1}$ or $\mathbf{1}_Y$ is the trivial holomorphic line bundle $Y \times \mathbb{C}$ over a manifold Y.

(NA.18) For a rank 2 holomorphic vector bundle $\mathcal{E}$ over Y we write $\mathbb{P}(\mathcal{E})$ for the $\mathbb{P}^1$-bundle that is the projectivization of $\mathcal{E}$.

(NA.19) If $\mathcal{L}$ is a holomorphic line bundle over Y and $\mathcal{E} = \mathcal{L} \oplus \mathbf{1}_Y$, then we call $\bar{\mathcal{L}} = \mathbb{P}(\mathcal{E})$ *the closure of* (*the total body of*) $\mathcal{L}$.

(NA.20) $\mathbb{C}(Z)$ stands for the field of rational functions on the irreducible complex projective variety Z.

(NA.21) Let X, Y be two compact connected irreducible reduced analytic complex spaces. A meromorphic map $f\colon X \to Y$ assigns to every point $x \in X$ a subset $f(x) \subset Y$ (the image of x) such that the following conditions are met:

(a) the graph $G_f := \{(x, y) \mid y \in f(x) \subset X \times Y\}$ is a connected irreducible complex analytic subspace of $X \times Y$ with $\dim(G_f) = \dim(X)$;

(b) there exists an open dense subset $X_0 \subset X$ such that $f(x)$ consists of one point for every $x \in X_0$.

(NA.22) The *general* point $x \in X$ is a point in an (analytically) Zariski open dense subset of X. The general fibre of a holomorphic map $f\colon X \to Y$ is the preimage $f^{-1}(y)$ of a general point $y \in Y$.

Definition 3.1. Following [25], we define a covering family of rational curves for a compact complex connected manifold X as a pair of morphisms $p\colon Z \to T$ and $\psi\colon Z \to X$ of compact irreducible complex spaces if the following conditions are satisfied:

- ψ is surjective;
- there is a dense analytical Zariski open subset $U \subset T$ such that for $t \in U$ the fibre $Z_t = g^{-1}(t)$ is isomorphic to $\mathbb{P}^1$ and $\dim(\psi(Z_t)) = 1$.

Manifolds X admitting a covering family with this property are called *uniruled* [25; Definition 2.1 and Lemma 2.2].

Remark 3.2. The Kodaira dimension $\kappa(X)$ equals $-\infty$ if X is a uniruled compact complex manifold [25; the remark on p. 691], [35; Corollary IV.1.11]. In low dimensions the converse is true.

Theorem 3.3 ([45] for projective manifolds, [29] for non-projective ones). *Let X be a compact Kähler manifold of dimension at most 3. Then X is uniruled if and only if $\kappa(X) = -\infty$.*

Remark 3.4 (Fujiki's theorems). It was proved by A. Fujiki for a compact connected complex manifold Y that

(i) if Y contains no rational curves, then for any complex manifold X every meromorphic map $f\colon X \to Y$ is holomorphic (see [24]);

(ii) $\operatorname{Aut}_0(Y)$ is isomorphic to a complex torus $\operatorname{Tor}(Y)$ (unless it is trivial) if Y is Kähler and either non-uniruled [23; Proposition 5.10] or with non-negative Kodaira dimension [23; Corollary 5.11].

The next statement (see [7; Proposition 1.4]) is similar to Lemma 3.1 of [34].

Proposition 3.5. *Let X be a connected complex compact Kähler manifold, and let $F = \operatorname{Aut}(X)/\operatorname{Aut}_0(X)$. Then the group F is bounded.*

Remark 3.6. Lemma 3.1 of J. H. Kim [34] states the following.

Let X be a normal compact Kähler variety. Then there exists a positive integer l, depending only on X, such that for any finite subgroup G of $\operatorname{Aut}(X)$ acting biholomorphically and meromorphically on X we have $[G : G \cap \operatorname{Aut}_0(X)] \leqslant l$.

We cannot use this lemma straightforwardly, since it is not clear why every finite subgroup of $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ should be isomorphic to $G/(G\cap \mathrm{Aut}_0(X))$ for some finite subgroup G of $\mathrm{Aut}(X)$.

COROLLARY 3.7. *Let X be a compact connected complex Kähler manifold that is either non-uniruled or with Kodaira dimension $\kappa(X)\geqslant 0$. Then $\mathrm{Aut}(X)$ is very Jordan.*

PROOF. In view of Proposition 3.5 it is sufficient to prove that $\mathrm{Aut}_0(X)$ is commutative. But this assertion follows from [23; Proposition 5.10] if X is non-uniruled and from [23; Corollary 5.11] if $\kappa(X)\geqslant 0$ (see Remark 3.4). The corollary is proved.

In general, let Z be a compact complex connected Kähler manifold. The analogue of the Chevalley decomposition for algebraic groups is valid for complex Lie group $\mathrm{Aut}_0(Z)$:

$$1\to L(Z)\to \mathrm{Aut}_0(Z)\to \mathrm{Tor}(Z)\to 1 \tag{2}$$

where $L(Z)$ is bimeromorphically isomorphic to a linear group, and $\mathrm{Tor}(Z)$ is a complex torus [23; Theorem 5.5], [40; Theorem 3.12], [16; Theorem 3.28].

REMARK 3.8. If $L(Z)$ in (2) is not trivial, Z contains a rational curve. Moreover, according to [23; Proposition 5.10], Z is bimeromorphic to a fibre space whose general fibre is $\mathbb{P}^1$, that is, X is *uniruled*.

Chapter 2. Rational bundles

In § 4, we want to persuade the reader that uniruled manifolds (in particular, $\mathbb{P}^1$-bundles) are of special interest from the point of view of Jordan properties. To this end we give a very brief and certainly non-complete overview of known facts in this field. In § 5 we provide some general properties of maps of manifolds endowed with fibration over a non-uniruled base with general fibre $\mathbb{P}^1$. In § 6 we deal with projective non-trivial conic bundles.

4. Uniruled vs non-uniruled: Jordan properties of the groups $\mathrm{Aut}(X)$, $\mathrm{Bim}(X)$, and $\mathrm{Bir}(X)$

In order to demonstrate the special role of uniruled manifolds from the point of view of Jordan properties, we present samples of results on the Jordan properties of $\mathrm{Aut}(X)$ and $\mathrm{Bim}(X)$ for various types of compact complex manifolds X.

The group $\mathrm{Aut}(X)$ is known to be Jordan if

- X is projective [44];
- X is a compact complex Kähler manifold [34];
- X is a compact complex space in Fujiki's class $\mathcal{C}$ ([43], also [66] for Moishezon threefolds).

REMARK 4.1. For the group $\mathrm{Aut}(X)$ "Jordan" implies "strongly Jordan" because *for every compact complex manifold X there is a constant $C=C(X)$ such that every finite subgroup $G\subset \mathrm{Aut}(X)$ may be generated by at most C elements.*

One can find the proof of this fact in [48; Theorem 1.3]. It is based on the same property for elementary abelian p-groups that was proved for a much wider class of topological spaces in [42], and on group-theoretic arguments (which, according to the author, had been explained to him by E. Khukhro and A. Jaikin). Thus this fact is also valid in a much more general situation.

Moreover, the connected identity component $\mathrm{Aut}_0(X)$ of $\mathrm{Aut}(X)$ is Jordan for every compact complex space X [61; Theorems 5 and 7]. An example of $X = E$, where E is an elliptic curve, shows that $\mathrm{Aut}(X)$ can be Jordan but not bounded. A classification of complex compact surfaces with bounded automorphisms group was done in [69].

As follows from Corollary 3.7, the group $\mathrm{Aut}(X)$ is very Jordan for any compact connected complex Kähler non-uniruled manifold X. For uniruled manifolds the situation changes: if $X = E \times \mathbb{P}^1$, then $\mathrm{Aut}(X) \cong \mathrm{PSL}(2,\mathbb{C}) \times \mathrm{Aut}(E)$ is neither bounded nor very Jordan.

The groups $\mathrm{Bir}(X)$ and $\mathrm{Bim}(X)$ of birational and bimeromorphic transformations, respectively, are more complicated. Low-dimensional cases are well understood. Consider the following *list*:

- E, an elliptic curve;
- A_n, an abelian variety of dimension n;
- S_b, a bielliptic surface;
- S_{K1}, a surface of Kodaira dimension 1;
- S_K, a Kodaira surface (it is not a Kähler surface).

Here are examples of results for low-dimensional cases.

(a) If X is a complex compact surface with non-negative Kodaira dimension, then $\mathrm{Bir}(X)$ is bounded unless it appears in the above list [67; Theorem 1.1].

(b) If X is a projective surface, then $\mathrm{Bir}(X)$ is Jordan if X is not birational to a product of an elliptic curve and $\mathbb{P}^1$ [58]. (The case of $X = \mathbb{P}^2$ was done earlier by J.-P. Serre [76].)

(c) If X is birational to a product of an elliptic curve and $\mathbb{P}^1$, then $\mathrm{Bir}(X)$ is not Jordan [85].

(d) If X is a projective threefold, then $\mathrm{Bir}(X)$ is not Jordan if and only if X is birational to a direct product $E \times \mathbb{P}^2$ or $S \times \mathbb{P}^1$, where S is a surface from the above list [65].

(e) The group $\mathrm{Bim}(X)$ is Jordan for any non-uniruled compact complex connected Kähler manifold of dimension 3 (see [70], [26]).

(f) If X is a non-algebraic uniruled compact Kähler threefold with non-Jordan group $\mathrm{Bim}(X)$, then X is bimeromorphic to $\mathbb{P}(\mathcal{E})$ for a holomorphic rank 2 vector bundle $\mathcal{E}$ on a two-dimensional complex torus S with $a(S) = 1$. Moreover, if $a(X) = 2$, then $X \approx S \times \mathbb{P}^1$ (see [68]).

The following theorem for complex projective varieties was proved by Yu. Prokhorov and C. Shramov (for $\dim(X) > 3$ under the assumption of the so-called *BAB-conjecture*, named after A. Borisov, L. Borisov, and V. Alexeev), and C. Birkar (who proved this conjecture) [62; Theorem 1.8], [10].

THEOREM 4.2. *Let X be a projective variety of dimension n. Then the following hold*:

(i) *the group* $\mathrm{Bir}(X)$ *has bounded finite subgroups provided that X is non-uniruled and has irregularity $q(X) = 0$;*

(ii) *the group* $\mathrm{Bir}(X)$ *is Jordan provided that* X *is non-uniruled*;
(iii) *the group* $\mathrm{Bir}(X)$ *is Jordan provided that* X *has irregularity* $q(X) = 0$.

Here $q(X) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X)$ is the irregularity of X. In particular, the Cremona group Cr_n of any rank n is Jordan [63]. The exact value of J_{Cr_2} is 7200 (E. Yasinsky [84]). The Jordan constant for $\mathrm{Bir}(X)$ for a rationally connected threefold X may be found in [64].

Let us sketch the proof of assertions (i) and (ii) of Theorem 4.2.

First, using the MMP (Minimal Model Program) the authors reduce the problem to considering the group $\mathrm{PAut}(X_m)$, where X_m is a special (relatively minimal) model of X and $\mathrm{PAut}(Z)$ stands for the group of birational self-maps of a variety Z that are isomorphisms in codimension 1. This means that $f \in \mathrm{PAut}(X_m)$ moves a divisor to a divisor and induces an automorphism $f_* = \psi(f)$ of the finitely generated abelian group $\mathrm{NS}^W(X_m) = \mathrm{Cl}(X_m)/\mathrm{Cl}^0(X_m)$, were $\mathrm{Cl}(X_m)$ stands for the group of Weil divisors on X_m modulo linear equivalence and $\mathrm{Cl}^0(X_m)$ consists of those Weil divisors that are algebraically equivalent to zero.

Thus there is a short exact sequence

$$0 \to G_i \xrightarrow{i} G \xrightarrow{\psi} \mathrm{Aut}(\mathrm{NS}^W(X)), \tag{3}$$

where $G_i = \ker(\psi)$ acts on each equivalence class in $\mathrm{Cl}(X_m)$. Since $\mathrm{NS}^W(X_m)$ is a finitely generated abelian group, $\mathrm{Aut}(\mathrm{NS}(X))$ is bounded.

Take a very ample divisor L and denote by $\mathrm{Cl}_L(X_m)$ the equivalence class containing L. It is an abelian variety of dimension $q(X_m) = q(X)$.

Let G_L be the kernel of the action of G_i on $\mathrm{Cl}_L(X_m)$. Then there is a short exact sequence

$$0 \to G_L \to G_i \to G_{ab}, \tag{4}$$

where $G_{ab} \subset \mathrm{Aut}(\mathrm{Cl}_L(X_m))$ is a subgroup of automorphisms (as a variety, but not as a group) of the abelian variety $\mathrm{Cl}_L(X_m)$. The group $\mathrm{Aut}(\mathrm{Cl}_L(X_m))$ is strongly Jordan. Let V be a linear space of sections of L and $\mathbb{P}(V)$ be its projectivization. Let F_L be the subgroup of those linear transformations of the projective space $\mathbb{P}(V)$ that preserve $X_m \subset \mathbb{P}(V)$. Since F_L is a linear group and X (and also X_m) is *non-uniruled*, F_L has to be finite (see Remark 3.8). Thus $G_L \subset F_L$ is finite.

Therefore,

- if $q(X) = 0$, then G_{ab} is trivial and $\mathrm{Bir}(X_m)$ is bounded (see Remark 2.9(i));
- if $q(x) > 0$, then G_i is Jordan (see Remark 2.9(iv)) and $\mathrm{Bir}(X)_m$ is Jordan (see Remark 2.9(iii)).

REMARK 4.3. (i) One can ask similar questions about the group $\mathrm{Diff}(M)$ of all diffeomorphisms of a smooth manifold M. There was the conjecture of E. Ghys (1997):

If M *is a compact smooth manifold*, *then* $\mathrm{Diff}(M)$ *is Jordan*.

It was answered negatively by B. Csikós, L. Pyber, E. Szabó in [19], whose approach was based on an algebraic geometry construction from [85] (see also Chapter 3 below).

In works of J. Winkelmann [83] and V. Popov [60] it was proved that there is a connected non-compact Riemann surface M such that $\mathrm{Aut}(M)$ contains an isomorphic copy of every finitely presented (in particular, every finite) group G.

In particular, Diff(M) is not Jordan. B. Zimmerman [88] proved that if M is compact and $\dim(M) \leqslant 3$, then Diff(M) is Jordan. The Jordan properties of Diff(M) were deeply studied by I. Mundet i Riera [47], [49]–[53]. It was proved there, in particular, that Diff(M) is Jordan if M is one of the following:

- an open acyclic manifold;
- a compact manifold (possibly with boundary) with non-zero Euler characteristic;
- a homology sphere.

(ii) The question on the Jordan properties of algebraic groups over various fields was considered in [61], [44], and [80] (see also [5]).

(iii) Jordan properties of Aut(X) and Bir(X) for open subsets of certain projective $\mathbb{P}^1$-bundles were considered in [4] and [6].

(iv) In the case of algebraic varieties X over algebraically closed fields of prime characteristic p one should not expect the Jordan properties to hold (see Example 2.7). However, there are analogues of several important results over $\mathbb{C}$ that deal instead with p-Jordan properties (see Remark 2.8) of Aut(X) and Bir(X) [30], [17], [36]. On the other hand, it is known that the Cremona group of rank 2 over a *finite* field is Jordan [71].

For compact complex manifolds, roughly speaking, from Jordan properties point of view the uniruled varieties are the worst and may be divided in several categories.

First, manifolds X that are rationally connected (or with $q(X) = 0$). For projective varieties, thanks to Theorem 4.2, Bir(X) is Jordan.

Second, manifolds that are fibred over a non-uniruled base Y with rationally connected fibres, with $q(X) \neq 0$, and that are not bimeromorphic (birational) to a direct product $Y \times \mathbb{P}^N$. In many special cases Bim(X) (or Bir(X)) is Jordan. Moreover, Aut(X) appears often to be very Jordan. We discuss some of these special cases in Chapter 4.

Third, X is isomorphic (bimeromorphic) to the direct product $Y \times \mathbb{P}^N$. If Y is a torus, and $a(Y) > 0$, then Bir(Y) is not Jordan. This case is the subject of Chapter 3.

5. Rational bundles

In this section we provide some useful facts about $\mathbb{P}^1$-bundles and their morphisms. We start with a slightly more general construction.

DEFINITION 5.1. We say that a triple (X, p, Y) is a *rational bundle* over Y if

(a) X, Y are compact connected complex manifolds endowed with a holomorphic surjective map $p\colon X \to Y$;

(b) for a general $y \in Y$ the fibre $p^*(y)$ is reduced and isomorphic to $\mathbb{P}^1$ (where *general* means "lying outside a proper analytic subset of Y"; see (NA.20) in § 3);

(c) If $\dim(P_y) = 1$ for every $y \in Y$, where $P_y := p^{-1}(y)$, then we call (X, p, Y) an *equidimensional rational bundle* over Y.

If for an open subset $U \subset Y$ and every $y \in U$ the fibre $P_y \sim \mathbb{P}^1$, then, by a theorem of W. Fischer and H. Grauert [22], $p^{-1}(U) \subset X$ is a holomorphically locally trivial fibre bundle over U. If $U = X$, then the triple (X, p, Y) is a $\mathbb{P}^1$-bundle over Y.

If (X, p, Y) is a rational bundle over a non-uniruled Kähler manifold Y, then $p\colon X \to Y$ is by definition a maximal rationally connected (MRC) fibration of X (for the definition and discussion, see [15; Theorem 2.3, Remark 2.8] and [35; IV.5]).

Bimeromorphic self-maps preserve the MRC fibration. This is a well-known fact, but we have not found a suitable reference for the proof of this fact in the complex analytic case. We provide it here. In the case when the Kodaira dimension satisfies $\kappa(Y) \geqslant 0$, the desired result follows from [41; Theorem 1.1.5]. For automorphisms the detailed exposition may be found in [1; § 2.4].

Lemma 5.2. *Let X, Y, and Z be three complex compact connected manifolds, and let $p\colon X \to Y$ and $q\colon X \to Z$ be surjective holomorphic maps. Assume that*

- *Z is non-uniruled;*
- *there is an analytical Zariski open dense subset $U \subset Y$ such that $P_u = p^{-1}(u) \sim \mathbb{P}^1$ for every $u \in U$.*

Then there is a meromorphic map $\tau\colon Y \dashrightarrow Z$ such that $\tau \circ p = q$, that is, the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y \arrow[rr, dashed, "\tau"] & & Z \end{tikzcd}$$

Proof. Let $\Phi\colon X \to Y \times Z$ be defined by $\Phi(x) = (p(x), q(x))$. The image $T = \Phi(X)$ is an irreducible compact analytic subspace of $Y \times Z$ (see, for example, [54; Chap. VII, Theorem 2]). We denote by pr_Y and pr_Z the natural projections of T onto the first and second factor, respectively. Both projections are evidently surjective. The set

$$T_1 = \{(y, z) \in T \mid \dim(\Phi^{-1}(y, z)) > 0\}$$

is an analytic subset of $T \subset Y \times Z$ ([73], [21; Theorem 3.6, p. 137]). Its projections $T_Y = \mathrm{pr}_Y(T_1) \subset Y$ and $T_Z = \mathrm{pr}_Z(T_1) \subset Z$ onto the first and the second factor are analytic subsets of Y and Z, respectively ([73], [54; Chap. VII, Theorem 2]).

If $T_Y \neq Y$, then $V := (Y \setminus T_Y) \cap U$ is an analytical Zariski open dense subset of Y. For each $y \in V$ we have $p^{-1}(y) \sim \mathbb{P}^1$ and $\dim(q(p^{-1}(y))) > 0$. Thus the pair $p\colon X \to Y$, $q\colon X \to Z$ would provide a covering family for Z, which is impossible, since Z is not uniruled. Thus $T_Y = Y$.

Take $u \in U$. Since $T_Y = Y$, there is $z \in Z$ such that

$$(u, z) \in T \quad \text{and} \quad \dim \Phi^{-1}(u, z) \geqslant 1.$$

Moreover,

$$\Phi^{-1}(u, z) = \{x \mid p(x) = u,\ q(x) = z\} \subset P_u \subset X.$$

Since $P_u \sim \mathbb{P}^1$ and $\dim(\Phi^{-1}(u, z)) \geqslant 1$, we have $P_u = \Phi^{-1}(u, z)$. Hence $q|_{P_u} = z$ for every $u \in U$ and some $z \in Z$ and there is only one $z \in Z$ such that $(u, z) \in T$.

Thus,

(a) T is an irreducible connected subset of $Y \times Z$;
(b) $\dim(T) = \dim(Y)$;
(c) for every $u \in U$ there is only one $z \in Z$ such that $(u, z) \in T$.

It follows that T is the graph of a meromorphic map, which we denote by τ.

Remark 5.3. The fact that q contracts every fibre of p over an analytical Zariski open non-empty subset of Y is proved in [27; Proposition 6.2].

Lemma 5.4. *Let (X, p_X, Y) and (W, p_W, Y) be two rational bundles over a non-uniruled (compact connected) manifold Y. Let $f\colon X \to W$ be a surjective meromorphic map.*

Then there exists a meromorphic map $\tau(f)\colon Y \to Y$ that can be included in the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_X"'] & W \arrow[d, "p_W"] \\ Y \arrow[r, "\tau(f)"] & Y \end{tikzcd} \tag{5}$$

In addition, if f is holomorphic, so is $\tau(f)$.

Proof. Let $a\colon \widetilde{X} \to X$ be a modification of X such that the following diagram is commutative:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "a"'] \arrow[dr, "b"] & \\ X \arrow[rr, dashed, "f"] & & W \end{tikzcd}$$

where $b\colon \widetilde{X} \to W$ is a holomorphic map (it always exists: see [56; Theorem 1.9]).

Consider the holomorphic maps

$$\tilde{p}_X := p_X \circ a\colon \widetilde{X} \to Y \quad \text{and} \quad \tilde{f} := p_W \circ b\colon \widetilde{X} \to Y.$$

We apply Lemma 5.2 to $\widetilde{X}$, $Y = Z$ and $\tilde{p}_X\colon \widetilde{X} \to Y$, $\tilde{f}\colon = \widetilde{X} \to Y$ and obtain the needed map $\tau(f) \in \mathrm{Bim}(Y)$ which can be included in the following commutative diagram:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "a"'] \arrow[dr, "b"] & \\ X \arrow[rr, dashed, "f"] \arrow[d, "\mathrm{pr}_X"] & & W \arrow[d, "\mathrm{pr}_W"] \\ X \arrow[rr, "\tau(f)"] & & Y \end{tikzcd}$$

If f is holomorphic, then one can take $\widetilde{X} = X$ and $U = Y$ (in the notation of Lemma 5.2). Then $\tau(f)$ will be defined at every point of Y.

Corollary 5.5. *For a rational bundle* (X, p, Y) *over a non-uniruled* (*complex connected compact*) *manifold* Y *there are natural group homomorphisms*

$$\tau\colon \operatorname{Aut}(X) \to \operatorname{Aut}(Y) \quad \textit{and} \quad \tilde{\tau}\colon \operatorname{Bim}(X) \to \operatorname{Bim}(Y)$$

such that

$$p \circ f = \tau(f) \circ p, \quad p \circ f = \tilde{\tau}(f) \circ p$$

for every $f \in \operatorname{Aut}(X)$ *or* $f \in \operatorname{Bim}(X)$, *respectively.*

Remark 5.6. If Y is Kähler non-uniruled, then the restriction group homomorphism

$$\tau|_{\operatorname{Aut}_0(X)}\colon \operatorname{Aut}_0(X) \to \tau(\operatorname{Aut}_0(X))$$

is a holomorphic homomorphism of complex Lie groups and $\tau(\operatorname{Aut}_0(X))$ is a closed complex Lie subgroup of $\operatorname{Aut}(Y)$ (A. Fujiki [23; Lemma 2.4, 3), Theorem 5.5, and Lemma 4.6]).

In what follows we use heavily the following classical theorems.

Theorem 5.7 (Remmert–Stein theorem; see, for example, [54; Chap. VII, Theorem of Remmert–Stein]). *Let* X *be a complex space*, Y *be an analytic subset of* X, *and* A *be an analytic subset of* $X \setminus Y$. *Suppose that there is an integer* $p > 0$ *such that* $\dim(Y) \leqslant p-1$, *while* $\dim_a(A) \geqslant p$ *for any* $a \in A$ ($\dim(Y) \leqslant -1$ *means that* $Y = \varnothing$). *Then the closure* $\bar{A}$ *of* A *in* X *is an analytic set in* X.

Theorem 5.8 (Riemann's second removable singularity theorem; [21; Chap. 2, Appendix]). *Assume that* X *is a complex manifold and* $A \subset X$ *is an analytic subset such that*

$$\operatorname{codim}_x(A) \geqslant 2 \quad \textit{for every } x \in X.$$

Then any holomorphic function $f\colon X \setminus A \to \mathbb{C}$ *has a unique holomorphic extension* $\tilde{f}\colon X \to \mathbb{C}$.

Theorem 5.9 (Levi's continuation theorem; [39], see also [54; Chap. VII, Theorem 4] or [21; § 4.8]). *Let* X *be a normal complex space and* Y *be an analytic subset of* X *such that for any* $a \in X$ *we have* $\dim_a(Y) \leqslant \dim_a(X) - 2$. *Then any meromorphic function on* $X \setminus Y$ *has an extension to a meromorphic function on* X.

Remark 5.10. It follows from the Riemann's second theorem that a holomorphic map from $f\colon X \setminus \Sigma \to Z$ where X is a complex manifold, Σ is an analytic subset of codimension at least 2, and $Z \subset \mathbb{C}^N$ is an affine complex set, can be extended to a holomorphic map $\tilde{f}\colon X \to Z$.

Indeed, let $z_1, \dots, z_N$ be coordinates in $\mathbb{C}^N$. The map f consists of N holomorphic functions $z_i(x)$, $i = 1, \dots, N$, defined on $X \setminus \Sigma$. By Theorem 5.8 the functions z_i can be extended to holomorphic functions $\tilde{z}_i$ defined on X. Since Z is a closed subset of $\mathbb{C}^N$, we have $\tilde{f}(x) = (\tilde{z}_1(y), \dots, \tilde{z}_N(x)) \in Z$ for every $x \in X$.

This fact is a particular case of the extension theorem due to A. Andreotti and W. Stoll [2]. Recall that a subset $M \subset X$ of a complex space X is *thin* if in a neighbourhood of every point $m \in M$ it is contained in an analytic subset of codimension 1.

THEOREM 5.11 (Andreotti–Stoll theorem). *Let $\tau\colon A \to Y$ be a holomorphic map of the open subset A of a normal complex space X into a Stein space Y. Let $M := X \setminus A$ be a thin set. If M has topological codimension at least 3, then τ can be extended to a holomorphic map of X into Y.*

We use this fact to prove the following lemma.

LEMMA 5.12. *Let (X, p, Y) and (Z, q, Y) be two $\mathbb{P}^1$-bundles over a connected complex manifold Y. Let $\Sigma \subset Y$ be an analytic subset of codimension at least 2, and let $U = Y \setminus \Sigma$, $V_X = p^{-1}(U)$, and $V_Z = q^{-1}(U)$. Let $f\colon X \to Y$ be a meromorphic map such that $q \circ f = p$ and the induced map $f\colon V_X \to V_Z$ is an isomorphism. Then $f\colon X \to Z$ is a biholomorphic isomorphism.*

PROOF. By construction, for every $u \in U$ the map f induces an isomorphism $f|_{P_y}\colon P_y \to Q_y$, where $P_y = p^{-1}(y)$ and $Q_y = q^{-1}(y)$. Consider a point $s \in \Sigma$ and an open neighbourhood U_s of it such that there are isomorphisms $\psi_X\colon p^{-1}(U_s) \to U_s \times \mathbb{P}^1$ and $\psi_Z\colon q^{-1}(U_s) \to U_s \times \mathbb{P}^1$ compatible with the projection maps p and q, respectively. Then for every $y \in U_s \cap U$ we have an element of $\mathrm{PSL}(2, \mathbb{C})$ representing $f|_{P_y}\colon P_y \to Q_y$, which is an automorphism of $\mathbb{P}^1$. Thus we have a holomorphic map $U_s \cap U \to \mathrm{PSL}(2, \mathbb{C})$. Since the target space is an affine set, this map extends to a holomorphic map $U_s \to \mathrm{PSL}(2, \mathbb{C})$. Hence we have an extension of f to an isomorphism $\tilde{f}_s\colon p^{-1}(U_s) \to q^{-1}(U_s)$, which coincides with f in $V_X \cap p^{-1}(U_s)$, hence everywhere.

LEMMA 5.13. *Let (X, p, Y) and (Z, q, Y) be two $\mathbb{P}^1$-bundles over a compact connected complex manifold Y with $\dim(Y) = n$. Let $\Sigma \subset Y$ be an analytic subset of codimension at least 2, and let $U = Y \setminus \Sigma$, $V_X = p^{-1}(U)$, and $V_Z = q^{-1}(U)$. Let $f\colon V_X \to V_Z$ be a meromorphic map such that $q \circ f = p$. Then there exists a meromorphic map $\tilde{f}\colon X \to Y$ such that $\tilde{f}\big|_U = f$ and $q \circ \tilde{f} = p$.*

For Kähler manifold Y this lemma is a consequence of the following general theorem of Y.-T. Siu [81].

THEOREM 5.14 (Siu's extension theorem). *Let X be a complex manifold, A be a subvariety of codimension at least 1 in X, and G be an open subset of X which intersects every branch of A of codimension 1. If M is a compact Kähler manifold, then every meromorphic map f from $(X - A) \cup G$ to M can be extended to a meromorphic map from X to M.*

At this stage we do not require that Y (and, *a fortiori*, Z) be Kähler, but we use the fact that X and Z are $\mathbb{P}^1$-bundles.

PROOF OF LEMMA 5.13. Consider the fibre product

$$W = X \times_Y Z = \{(x, z) \in X \times Z \mid p(x) = q(z)\} \subset X \times Z$$

and its subsets

$$\Gamma_f = \{(x, z) \in V_X \times V_Z \mid p(x) = q(z),\ z \in f(x)\} \subset W,$$
$$\widetilde{\Sigma} = \{(x, z) \in X \times Z \mid p(x) = q(z) \in \Sigma\} \subset W.$$

By construction $\dim(\widetilde{\Sigma}) \leqslant n$ and $\dim(\Gamma_f) = \dim(X) = n+1$. Thus, according to the Remmert–Stein theorem (Theorem 5.7) the closure $\overline{\Gamma}_f$ of Γ_f in W is an analytic subset of W. Let $U_1 \subset U$ be an open subset such that f is defined at every point of $V_1 := p^{-1}(U_1)$. We have

- $\overline{\Gamma}_f$ is an irreducible (since Γ_f, being the graph of a meromorphic map, is irreducible) analytic subset of $X \times Z$;
- $\dim(\overline{\Gamma}_f) = \dim(X)$;
- for every $v \in V_1$ there is unique $z \in Z$ such that $(v, z) \in \overline{\Gamma}_f$;
- the natural projection $\tau\colon \overline{\Gamma}_f \to X$ is proper, since both sets are compact.

It follows that $\overline{\Gamma}_f$ is a graph of a meromorphic map $\tilde{f}\colon X \to Z$ (see [3; p. 75]).

We also use the following lemma.

Lemma 5.15. *Assume that Y is a compact connected complex manifold, $\Sigma \subset Y$ is an analytic subset of codimension at least 2, and let $U = Y \setminus \Sigma$. Let $(\mathcal{L}, \pi, Y)$ be a holomorphic line bundle over Y such that $\mathcal{L}|_U$ is trivial. Then $\mathcal{L}$ is trivial.*

Proof. Indeed, $V := \pi^{-1}(U) \sim U \times \mathbb{C}_z$, thus $z = F(v)$ is a holomorphic function on V. The set $\widetilde{\Sigma} := \pi^{-1}(\Sigma)$ has codimension at least 2 in $\mathcal{L}$. By Riemann's second removable singularity theorem (Theorem 5.8) F can be extended to a holomorphic function $\overline{F}$ on $\mathcal{L}$. Thus we have a holomorphic map $\Phi\colon \mathcal{L} \to Y \times \mathbb{C}_z$, $x \in \mathcal{L} \to (p(x), \overline{F}(x))$, which is an isomorphism outside $\widetilde{\Sigma}$. Let S be the set of all points in $\mathcal{L}$ where the differential $d\Phi$ of Φ does not have the maximum rank. The sets S and $\widetilde{S} = p(S)$ are analytic subsets of $\mathcal{L}$ and Y, respectively (see, for instance, [54; Chap. VII, Theorem 2], [57; Theorem 1.22], and [73]). Moreover, $\operatorname{codim}(\widetilde{S}) = 1$ (see [72]). But $\widetilde{S} \subset \Sigma$, hence $\widetilde{S} = \varnothing$. It follows that Φ is an isomorphism.

6. Non-trivial rational bundles

In this section we consider non-trivial $\mathbb{P}^1$-bundles over a non-uniruled base. It appears that the fact that $X \not\approx Y \times \mathbb{P}^1$ imposes significant restrictions on the structure of the groups $\operatorname{Aut}(X)$ and $\operatorname{Bim}(X)$. We start with the projective case.

Definition 6.1. A regular surjective map $f\colon X \to Y$ of smooth irreducible projective complex varieties is a *conic bundle* over Y if there is a Zariski open dense subset $U \subset Y$ such that the fibre $f^{-1}(y) \sim \mathbb{P}^1$ for all $y \in U$.

The *generic fibre* of f is an irreducible smooth projective curve $\mathcal{X}_f$ over the field $K := \mathbb{C}(Y)$ such that its field of rational functions $K(\mathcal{X}_f)$ coincides with $\mathbb{C}(X)$. (The genus of $\mathcal{X}_f$ is 0.)

Theorem 6.2 [5]. *Let X be a conic bundle over a non-uniruled smooth irreducible projective variety Y with $\dim(Y) \geqslant 2$. If X is not birational to $Y \times \mathbb{P}^1$, then $\operatorname{Bir}(X)$ is strongly Jordan.*

Let us sketch the proof of Theorem 6.2.

Let $f\colon X \to Y$ be a conic bundle and assume that Y is non-uniruled. According to Corollary 5.5, every $\phi \in \mathrm{Bir}(X)$ is fibrewise: there is a homomorphism $\tilde{\tau}\colon \mathrm{Bir}(X) \to \mathrm{Bir}(Y)$ such that $\tilde{\tau}(\phi) \circ f = f \circ \phi$:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\tilde{\tau}(\phi)"] & Y \end{tikzcd}$$

It follows that there is an exact sequence of groups

$$0 \to \mathrm{Bir}_{\mathbb{C}(Y)}(\mathcal{X}_f) \to \mathrm{Bir}(X) \to \mathrm{Bir}(Y). \tag{6}$$

Since Y is non-uniruled, the group $\mathrm{Bir}(Y)$ is strongly Jordan thanks to Theorem 4.2 (see also [5; Corollary 3.8 and its proof]).

Let us compute $\mathrm{Bir}_K(\mathcal{X}_f)$. We have:

1) $\mathrm{Bir}_K(\mathcal{X}_f) = \mathrm{Aut}(\mathcal{X}_f)$ since $\dim(\mathcal{X}_f) = 1$;

2) since $X \not\approx Y \times \mathbb{P}^1$, the genus 0 curve $\mathcal{X}_f$ has *no* K-points and therefore there exists a ternary quadratic form

$$q(T) = a_1T_1^2 + a_2T_2^2 + a_3T_3^2$$

over K such that

(a) all the a_i are non-zero elements of K,

(b) $q(T) = 0$ if and only if $T = (0, 0, 0)$ (this means that q is *anisotropic*),

(c) $\mathcal{X}_f$ is biregular over K to the plane projective quadric

$$\mathbf{X}_q := \{(T_1 : T_2 : T_3) \mid q(T) = 0\} \subset \mathbb{P}^2_K;$$

3) K is a field of characteristic zero that contains all roots of unity.

Now we consider a quadric, that is, a hypersurface in a projective space defined by one irreducible quadratic equation over K. It is anisotropic if it has no point defined over K. The following theorem was proved in [5].

THEOREM 6.3 [5]. *Suppose that K is a field of characteristic zero that contains all roots of unity, let $d \geqslant 3$ be an odd integer, V be a d-dimensional K-vector space, and $q\colon V \to K$ be a quadratic form such that $q(v) \neq 0$ for all non-zero $v \in V$. Consider the projective quadric $X_q \subset \mathbb{P}(V)$ defined by the equation $q = 0$, which is a smooth projective irreducible $(d-2)$-dimensional variety over K. Let $\mathrm{Aut}(X_q)$ be the group of biregular automorphisms of X_q. Let G be a finite subgroup of $\mathrm{Aut}(X_q)$. Then G is commutative, all its non-identity elements have order 2, and the order of G divides 2^{d-1}.*

Thus, if G is a non-trivial finite subgroup of $\mathrm{Aut}(\mathcal{X}_f)$, then either $G \cong \mathbb{Z}/2\mathbb{Z}$ or $G \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Now applying Remark 2.9(iv), from (6) we get that $\mathrm{Bir}(X)$ is Jordan.

REMARK 6.4. Actually, in Theorem 6.2 the variety X is considered as a pointless ($X(K) = \varnothing$) rational curve defined over a field K, where K contains all roots of unity. "Pointless surfaces" were studied by C. Shramov and V. Vologodsky in [79] and [80].

For complex compact manifolds the absence of points in the generic fibre has to be reformulated in terms of sections.

Let (X, p, Y) be a rational bundle over a compact complex connected non-uniruled manifold Y (see Definition 5.1), that is,

- X and Y are compact connected manifolds;
- Y is non-uniruled;
- $p\colon X \to Y$ is a surjective holomorphic map;
- $p^{-1}(U)$ is a holomorphic locally trivial fibre bundle over a dense analytical Zariski open subset $U \subset Y$ with fibre $\mathbb{P}^1$ and with the corresponding projection map $p\colon p^{-1}(U) \to U$.

According to Lemma 5.4, every map $f \in \mathrm{Bim}(X)$ maps the general fibre of p to a fibre of p. Let

$$\mathrm{Aut}(X)_p = \{f \in \mathrm{Aut}(X) \mid \tau(f) = \mathrm{id}\}, \qquad \mathrm{Bim}(X)_p = \{f \in \mathrm{Bim}(X) \mid \tilde{\tau}(f) = \mathrm{id}\}$$

be the kernels of τ and $\tilde{\tau}$, respectively.

Then we have the following short exact sequences:

$$0 \to \mathrm{Aut}(X)_p \to \mathrm{Aut}(X) \xrightarrow{\tau} \mathrm{Aut}(Y), \tag{7}$$

$$0 \to \mathrm{Bim}(X)_p \to \mathrm{Bim}(X) \xrightarrow{\tilde{\tau}} \mathrm{Bim}(Y). \tag{8}$$

Definition 6.5. Let (X, p, Y) be an equidimensional rational bundle over a compact complex connected non-uniruled manifold Y. We call an irreducible analytic subspace D of X an *almost section* if the intersection number (D, F) of D with a fibre $F = p^{-1}(y)$, $y \in Y$, is 1.

Remark 6.6. For $f \in \mathrm{Bim}(X)_p$ let $\widetilde{S}_f$ be the indeterminacy locus of f, which is an analytic subspace of X of codimension at least 2 [73; p. 369]. Let $S_f = p(\widetilde{S}_f)$, which is an analytic subset of Y [73], [54; Chap. VII, Theorem 2]. Since the dimension of a fibre of p is one, $Y \setminus S_f$ is an analytical Zariski open dense subset U of Y. Hence the restriction $f|_{P_y}$ of f to the fibre $P_y = p^{-1}(y)$ of p over a general point $y \in Y$ belongs to $\mathrm{Aut}(P_y)$. Thus f induces an automorphism of $V = p^{-1}(U)$ onto itself.

Let D be an almost section of X.

(i) Let $a\colon \widetilde{X} \to X$ be such a modification of X that the following diagram is commutative:

$$\begin{tikzcd} & X \arrow[dl, "a"'] \arrow[dr, "b"] & \\ X \arrow[rr, dashed, "f"] & & X \end{tikzcd}$$

where $b\colon \widetilde{X} \to X$ is a holomorphic map (it always exists [56; Theorem 1.9]). Then $f(D) = ba^{-1}(D)$ is an analytic subset ([73], [21; Theorem 3.6]) which is a union of finite number of irreducible components $D_1, \ldots, D_n$.

(ii) We may assume (maybe after shrinking U) that D meets every fibre P_y, $y \in U$, at precisely one point. Then $f(D)$ also meets P_y, $y \in U$, at precisely one point.

(iii) It follows from (ii) that precisely one irreducible component of $f(D)$, say, D_1, meets a fibre P_y, $y \in U$. The intersection $D_1 \cap P_y$, $y \in U$, consists of a unique point.

Thus D_1 is an almost section. It follows that the image of an almost section under $f \in \mathrm{Bim}(X)_p$ contains precisely one almost section. In particular, f cannot contract an almost section.

Similarly, if $\Phi\colon X \to Z$ is a bimeromorphic map of a $\mathbb{P}^1$-bundle (X, p, Y) to a $\mathbb{P}^1$-bundle (Z, q, Y) such that $q \circ \Phi = p$, then the image of an almost section contains an almost section.

The following results were proved by Yu. Prokhorov and C. Shramov in a more general setting. We formulate below an application of these results to the case of $\mathbb{P}^1$-bundles.

THEOREM 6.7. *Let (X, p, Y) be a $\mathbb{P}^1$-bundle over a compact complex connected non-uniruled manifold Y. Let $P_y = p^{-1}(y)$ be a fibre of p over a general point $y \in Y$. Then the following hold.*

1) *Every countable union of finite subgroups of* $\mathrm{Bim}(X)_p$ *can be embedded into* $\mathrm{Bim}(P_y)$ [68; Lemma 4.1].
2) *If X is Kähler, then* $\mathrm{Bim}(X)_p$ *is Jordan* [68; Corollary 4.3].
3) *If there exists an almost section D on X, then $X \sim \mathbb{P}(\mathcal{E})$ for some rank* 2 *holomorphic vector bundle $\mathcal{E}$ on Y* [78; Lemma 3.5].
4) *Assume that no almost section exists on X. Assume that* $\mathrm{Bim}(Y)$ *is strongly Jordan. Then* $\mathrm{Bim}(X)$ *is Jordan* [78; Corollary 5.8].
5) *If there exists $f \in \mathrm{Bim}(X)_p$ of finite order $d > 2$, then there exist at least two distinct almost sections on X. If f is biholomorphic, these almost sections can be chosen to be disjoint.* (*See* [78; Lemma 4.1].)

To this we add the following lemma.

LEMMA 6.8. *In the notation of Theorem 6.7, assume that there exists precisely one almost section on X. If* $\mathrm{Bim}(Y)$ *is Jordan, so is* $\mathrm{Bim}(X)$.

PROOF. Assume that D is the only almost section. Let $f \in \mathrm{Bim}(X)_p$, $f \neq \mathrm{id}$. The set $f(D)$ contains an irreducible component D_1 that is an almost section (see Remark 6.6). Therefore, $D = D_1$ and D is contained in the set $\mathrm{Fix}(f)$ of fixed points of f. Let $V \subset Y$ be an analytical Zariski open dense subset such that the restriction f_v of f to the fibre P_v is a non-identical automorphism of P_v for all $v \in V$. Since f_v has at most two fixed points, we have the following alternative:

- either $\mathrm{Fix}(f) \cap P_v = D \cap P_v$ contains one point and f_v has infinite order;
- or $(\mathrm{Fix}(f) \cap P_v) \setminus (D \cap P_v)$ contains a point for the general $v \in V$ and $\mathrm{Fix}(f)$ contains an almost section distinct from D, which is impossible.

Thus, every element $f \in \mathrm{Bim}(X)_p$ different from id has infinite order. Therefore, $G \cap \mathrm{Bim}(X)_p = \{\mathrm{id}\}$ for every finite group $G \subset \mathrm{Bim}(X)$ and $\tilde{\tau}\colon G \to \mathrm{Bim}(Y)$ is a group embedding. Hence the Jordan index $J_{\mathrm{Bim}(X)} \leqslant J_{\mathrm{Bim}(Y)}$.

The opposite case, when the $\mathbb{P}^1$-bundle has many almost sections, is when $X \cong Y \times \mathbb{P}^1$. It is considered in the next chapter.

Chapter 3. $\mathbb{P}^1$-bundles over complex tori

In this chapter we deal with $\mathbb{P}^1$-bundles of special type, namely, $(\bar{\mathcal{L}}, p, T)$, where $\mathcal{L}$ is a holomorphic line bundle over a complex torus T and $\bar{\mathcal{L}} = \mathbb{P}(\mathcal{L} \oplus \mathbf{1}_T)$. Most examples of compact complex connected manifolds with non-Jordan group $\mathrm{Bim}(X)$ (at least for dimensions greater than 3) are $\mathbb{P}^1$-bundles of this type. Manifolds of this type were studied by one of the authors in [85] (the projective case) and [87] (the non-algebraic case). The goal of this chapter is to present a unified approach for both situations. It is based on a construction motivated by symplectic geometry and inspired by an algebraic approach to theta functions developed in [46]. The chapter starts with symplectic constructions, then theta groups follow, and then we arrive at the description of certain subgroups of $\mathrm{Bim}(\bar{\mathcal{L}})$.

7. Symplectic group theory

This section contains elementary but useful facts about Jordan properties of central extensions of commutative groups by $\mathbb{C}^*$.

Consistent with the tradition, some groups are written in the multiplicative and some in the additive form. We hope that no confusion will arise.

Definition 7.1. A *symplectic pair* is a pair (A, e) that consists of a commutative group A and an alternating bilinear pairing

$$e\colon A \times A \to \mathbb{C}^*.$$

Here *alternating* means that

$$e(a,a) = 1 \quad \forall a \in A.$$

Bilinearity means that

$$\begin{aligned} e(a_1 + a_2, b) &= e(a_1, b)e(a_2, b), \\ e(a, b_1 + b_2) &= e(a, b_1)e(a, b_2) \end{aligned} \qquad \forall a, a_1, a_2, b, b_1, b_2 \in A.$$

These properties imply that for all $a, b \in A$

$$1 = e(a+b, a+b) = e(a,a)e(a,b)e(b,a)e(b,b) = e(a,b)e(b,a),$$

that is,

$$e(a,b) = e(b,a)^{-1} \quad \forall a, b \in A.$$

As usual, e gives rise to the group homomorphism

$$\Psi_e\colon A \to \mathrm{Hom}(A, \mathbb{C}^*), \quad b \mapsto \{\Psi_e(b)\colon A \to \mathbb{C}^*,\ a \mapsto e(a,b)\}. \tag{9}$$

A subgroup B of A is called *isotropic* with respect to e if

$$e(B, B) = \{1\}.$$

We define the *kernel* of e by

$$\ker(e) := \{a \in A \mid e(a, A) = \{1\}\} = \ker(\Psi_e);$$

this is a subgroup of A, which is isotropic with respect to e.

We say that e is *non-degenerate* if $\ker(e) = \{0\}$, that is,

$$\Psi_e \colon A \to \mathrm{Hom}(A, \mathbb{C}^*)$$

is an *injective* homomorphism. If e is non-degenerate, then we call (A, e) a *non-degenerate symplectic pair.*

EXAMPLE 7.2. Let d be a positive integer, let $\mathbf{S}_d = (\frac{1}{d}\mathbb{Z}/\mathbb{Z})^2 \cong (\mathbb{Z}/d\mathbb{Z})^2$, and let

$$\mathbf{e}_d \colon \mathbf{S}_d \times \mathbf{S}_d \to \mathbb{C}^*, \quad (a_1 + \mathbb{Z}, b_1 + \mathbb{Z}), (a_2 + \mathbb{Z}, b_2 + \mathbb{Z}) \mapsto \exp(2\pi \mathbf{i} d(a_1 b_2 - a_2 b_1)).$$

Then $(\mathbf{S}_d, \mathbf{e}_d)$ is a *non-degenerate symplectic pair.*

REMARK 7.3. Let $(A_1 e_1)$ and (A_2, e_2) be non-degenerate symplectic pairs. Consider the bilinear alternating form

$$\begin{aligned} e_1 e_2 \colon (A_1 \oplus A_2) \times (A_1 \oplus A_2) &\to \mathbb{C}^*, \\ (a_1, a_2), (b_1, b_2) &\mapsto e_1(a_1, b_1) \cdot e_2(a_2, b_2). \end{aligned}$$

Then $(A_1 \oplus A_2, e_1 e_2)$ is a *non-degenerate symplectic pair.*

REMARK 7.4. If (A, e) is a symplectic pair and B is a subgroup of A, then $(B, e|_B)$ is also a symplectic pair. Here $e|_B$ is the restriction of e to $B \times B$.

REMARK 7.5. (i) Each symplectic pair (A, e) gives rise to a non-degenerate symplectic pair $(\bar{A}, \bar{e})$, where

$$\bar{A} = A/\ker(e), \qquad \bar{e}(a \ker(e), b \ker(e)) = e(a, b) \quad \forall a, b \in A. \tag{10}$$

(ii) Clearly, a subgroup B of A is isotropic with respect to e if and only if its image $\bar{B}$ in $\bar{A}$ is isotropic with respect to $\bar{e}$. In particular, B is isotropic if and only if $B + \ker(e)$ is isotropic.

(iii) Let B be a subgroup of A. One can restate the property of B to be isotropic with respect to e as follows. The composition of $\Psi_e \colon A \to \mathrm{Hom}(A, \mathbb{C}^*)$ with the *restriction map* $\mathrm{Hom}(A, \mathbb{C}^*) \to \mathrm{Hom}(B, \mathbb{C}^*)$ is a group homomorphism

$$A \xrightarrow{\Psi_e} \mathrm{Hom}(A, \mathbb{C}^*) \to \mathrm{Hom}(B, \mathbb{C}^*). \tag{11}$$

Clearly, the kernel $B^\perp$ of this homomorphism (which is the *orthogonal complement* of B in A with respect to e) contains B if and only if B is isotropic.

(iv) Suppose that B coincides with $B^\perp$. This means that if $a \in A \setminus B$, then $e(B, a) \neq \{1\}$. In other words, B is a *maximal* isotropic subgroup of A with respect to e.

Conversely, suppose that B is a *maximal* isotropic subgroup of A with respect to e. Since B is isotropic, it follows that

$$B \subset B^\perp \subset A, \quad e(B^\perp, B) = \{1\}.$$

If $B^\perp \neq B$, then there is $a \in B^\perp \setminus B$ such that $e(a, B) = \{1\}$. This implies that the subgroup B_1 of A generated by B and a is isotropic, which contradicts the maximality of B.

It follows that $B = B^\perp$ if and only if B is a *maximal isotropic* subgroup of A.

Remark 7.6. Suppose that A is finite. Then the finite groups A and $\mathrm{Hom}(A, \mathbb{C}^*)$ are (non-canonically) isomorphic; in particular, they have the same order. It follows that in the case of finite A the pairing e is non-degenerate if and only if Ψ_e is a group *isomorphism*.

Lemma 7.7 (useful lemma). *Let (A, e) be a symplectic pair such that $A/\ker(e)$ is a finite group. If B is a maximal isotropic subgroup of A, then the index $[A : B]$ equals $\sqrt{\#(A/\ker(e))}$. In particular, if e is non-degenerate, then*

$$[A : B] = \sqrt{\#(A)} = \#(B).$$

Proof. In light of Remark 7.5, B contains $\ker(e)$ and therefore it suffices to prove the desired result for non-degenerate $(\bar{A}, \bar{e})$ (instead of (A, e)). In other words, without loss of generality, we may assume that $\ker(e) = \{0\}$, that is, $A = \bar{A}$ is *finite* and $e = \bar{e}$ is *non-degenerate*.

Since $\mathbb{C}^*$ is a divisible group, every group homomorphism $B \to \mathbb{C}^*$ extends to a group homomorphism $A \to \mathbb{C}^*$. This means that the restriction map

$$\mathrm{Hom}(A, \mathbb{C}^*) \to \mathrm{Hom}(B, \mathbb{C}^*)$$

is surjective. Since A is finite, the non-degeneracy of e means (in light of Remark 7.6) that $\mathrm{Hom}(A, \mathbb{C}^*) = \Psi_e(A)$. On the other hand the maximality of B means that the kernel of the *surjective* composition

$$A \overset{\Psi_e}{\cong} \mathrm{Hom}(A, \mathbb{C}^*) \twoheadrightarrow \mathrm{Hom}(B, \mathbb{C}^*)$$

coincides with B (see Remark 7.5), and therefore there is an *injective* group homomorphism

$$A/B \hookrightarrow \mathrm{Hom}(B, \mathbb{C}^*),$$

which is also surjective and therefore is an isomorphism. This implies that

$$\#(A/B) = \#\big(\mathrm{Hom}(B, \mathbb{C}^*)\big) = \#(B),$$

which finishes the proof if we take into account that $\#(A/B) = \#(A)/\#(B)$.

Remark 7.8. Suppose that $\ker(e)$ is either *finite* or *divisible*. Then every finite subgroup $\bar{B}$ of $\bar{A}$ is the image of a finite subgroup $B \subset A$ under $A \twoheadrightarrow \bar{A}$. Indeed, if $\ker(e)$ is finite, then one can take as B the preimage of $\bar{B}$ in A. If $\ker(e)$ is divisible, then it is a direct summand of A, that is, A splits into a direct sum $A = \ker(e) \oplus A'$ and the map $A \to \bar{A}$ induces an isomorphism $A' \cong \bar{A}$. Now one can take as B the (isomorphic) preimage of $\bar{B}$ in A'.

Definition 7.9. A symplectic pair (A, e) is called *almost isotropic* if there exists a positive integer D that enjoys the following property.

Each finite subgroup $\mathcal{B}$ of A contains an *isotropic* (with respect to e) subgroup $\mathcal{A}$ such that the index $[\mathcal{B} : \mathcal{A}] \leqslant D$. The smallest D with this property is called the *isotropy defect* of (A, e) and denoted by $D_{A,e}$.

Example 7.10. If $e \equiv 1$, then every subgroup is isotropic, and therefore $D_{A,e} = 1$.

REMARK 7.11. Suppose that $\ker(e)$ is either finite or divisible.

(i) It follows from Remarks 7.8 and 7.5 that (A, e) is almost isotropic if and only if $(\bar{A}, \bar{e})$ is almost isotropic. In addition, if this is the case, then

$$D_{A,e} = D_{\bar{A},\bar{e}}. \tag{12}$$

Indeed, let $\mathcal{A}$ be a finite subgroup of A and B be an isotropic subgroup of largest possible order in $\mathcal{A}$. In particular, B is a maximal isotropic subgroup of $\mathcal{A}$. Since $B_1 = B+(\mathcal{A}\cap\ker(e))$ is an isotropic subgroup of $\mathcal{A}$ that contains B, the maximality of B implies that $B_1 = B$, that is, $B \supset \mathcal{A} \cap \ker(e)$. This implies that the index $(\mathcal{A} : B)$ equals the index $[\bar{\mathcal{A}} : \bar{B}]$ where the subgroups $\bar{\mathcal{A}}$ and $\bar{B}$ are the images in $\bar{A}$ of $\mathcal{A}$ and B, respectively. Taking into account that $\bar{B}$ is an isotropic (with respect to $\bar{e}$) subgroup of the finite group $\bar{\mathcal{A}} \subset \bar{A}$, we conclude that

$$D_{A,e} \geqslant D_{\bar{A},\bar{e}}.$$

Conversely, suppose that $\bar{B}$ is an isotropic (with respect to $\bar{e}$) subgroup of maximum order of a finite group $\bar{\mathcal{A}} \subset \bar{A}$. As above, this implies that $\bar{B}$ is a maximal isotropic subgroup of $\bar{\mathcal{A}}$. By Remark 7.8, A contains a finite subgroup $\mathcal{A}$ whose image in $\bar{A}$ coincides with $\bar{\mathcal{A}}$. Let B the preimage of $\bar{B}$ in $\mathcal{A}$. Then B is isotropic with respect to e and the index $[\mathcal{A} : B]$ coincides with the index $[\bar{\mathcal{A}} : \overline{B}]$. This implies that

$$D_{A,e} \leqslant D_{\bar{A},\bar{e}},$$

which ends the proof.

(ii) Assume additionally that $\bar{A}$ is *finite*. Applying Lemma 7.7 to subgroups of $\bar{A}$ and using (12) we conclude that

$$D_{A,e} = D_{\bar{A},\bar{e}} = \sqrt{\#(\bar{A})}. \tag{13}$$

DEFINITION 7.12. A *theta group* attached to a symplectic pair (A, e) is a group G that sits in a short exact sequence

$$1 \to \mathbb{C}^* \xrightarrow{i} G \xrightarrow{j} A \to 0 \tag{14}$$

that enjoys the following properties.

The image of $\mathbb{C}^*$ is a *central* subgroup of G, and the alternating *commutator* pairing

$$A \times A \to \mathbb{C}^*, \qquad j(g_1), j(g_2) \mapsto i^{-1}(g_1 g_2 g_1^{-1} g_2^{-1}) \in \mathbb{C}^* \quad \forall g_1, g_2 \in G,$$

attached to the exact sequence (14) coincides with e.

REMARK 7.13. Every *central* extension G of a commutative group A by $\mathbb{C}^*$ gives rise to the symplectic pair (A, e) where $e(a_1, a_2) \in \mathbb{C}^*$ is the commutator of the preimages of a_1 and a_2 in G (for all $a_1, a_2 \in A$). This makes G a theta group attached to (A, e).

REMARK 7.14. (i) Clearly, an element g of the theta group G lies in the centre of G if and only if

$$e(j(g), j(h)) = 1 \quad \forall h \in G.$$

Since $j(G) = A$, the element g is central if and only if $j(g) \in \ker(e)$. This implies that the centre of G coincides with $j^{-1}(\ker(e))$.

(ii) Clearly, a subgroup H of G is commutative if and only if its image $j(H) \subset A$ is an isotropic subgroup of A with respect to e.

Remark 7.15. Let G be a theta group that sits in the short exact sequence (14). If B is a subgroup of A, then obviously the preimage $j^{-1}(B)$ is a theta group attached to the symplectic pair $(B, e|_B)$.

Lemma 7.16. *Let B be a finite subgroup of A. Then there exists a finite subgroup $\widetilde{B}$ of the theta group G such that $j(\widetilde{B}) = B$.*

Proof. In what follows we identify $\mathbb{C}^*$ with its image in G and view it as a certain central subgroup of G. Let d be the *exponent* of B.

Consider the finite multiplicative subgroups μ_d and μ_{d^2} of all dth roots of unity and d^2th roots of unity, respectively, in $\mathbb{C}^*$. We have

$$\mu_d \subset \mu_{d^2} \subset \mathbb{C}^* \subset G;$$

in addition,

$$e(B, B) \subset e(B, A) \subset \mu_d. \tag{15}$$

For every $b \in B$ choose its *lifting* $\tilde{b} \in G$ such that

$$\tilde{b}^d = 1, \quad \tilde{b}^{-1} = \widetilde{b^{-1}} \quad \forall b \in B; \tag{16}$$

this is possible since $\mathbb{C}^*$ is a central divisible subgroup of $\mathbb{C}^*$. Indeed, let $\tilde{b}_1 \in G$ be any lifting of b to G, that is, $j(\tilde{b}_1) = b$. Then

$$z_1 := \tilde{b}_1^d \in \ker(j) = \mathbb{C}^*.$$

We choose any

$$z = \sqrt[d]{z_1} \in \mathbb{C}^*$$

and put $\tilde{b} = z^{-1}\tilde{b}_1 \in G$. We have

$$j(\tilde{b}) = j(z^{-1}) + j(\tilde{b}_1) = 0 + b = b, \qquad \tilde{b}^d = (z^{-1})^d\, \tilde{b}_1^d = z_1^{-1} z_1 = 1.$$

Set

$$\widetilde{B} := \{\gamma\tilde{b} \mid \gamma \in \mu_{d^2},\, b \in B\} \subset G.$$

Clearly, $\widetilde{B}$ is finite, $j(\widetilde{B}) = B$, and

$$1 \in \mu_{d^2} \subset \widetilde{B} = \widetilde{B}^{-1} := \{u^{-1} \mid u \in \widetilde{B}\}$$

(the latter equality follows from the invariance of the central subgroup μ_{d^2} and the subset $\{\tilde{b} \mid b \in B\}$ under the map $u \mapsto u^{-1}$).

So, in order to prove that $\widetilde{B}$ is a subgroup of G, it suffices to check that $\widetilde{B}$ is closed under multiplication in G. Let $b_1, b_2 \in B$ and $b_3 = b_1 + b_2 \in B$. We need to compare $\tilde{b}_1\tilde{b}_2$ and $\tilde{b}_3$ in G. Clearly, there is $\gamma \in \mathbb{C}^*$ such that

$$\tilde{b}_3 = \gamma\tilde{b}_1\tilde{b}_2.$$

Notice that

$$\tilde{b}_1^d = \tilde{b}_2^d = \tilde{b}_3^d = 1 \in \mathbb{C}^* \subset G.$$

On the other hand, in light of (15),

$$\gamma_0 := \tilde{b}_1 \tilde{b}_2 \tilde{b}_1^{-1} \tilde{b}_2^{-1} = e(b_1, b_2) \in \mu_d \subset \mathbb{C}^* \subset G.$$

It follows that the images of $\tilde{b}_1$ and $\tilde{b}_2$ in the quotient G/μ_d commute, and therefore the image of $\tilde{b}_1\tilde{b}_2$ in G/μ_d has an order which divides d. This means that

$$\left(\tilde{b}_1\tilde{b}_2\right)^d \in \mu_d,$$

and therefore

$$\left(\tilde{b}_1\tilde{b}_2\right)^{d^2} = 1.$$

It follows that

$$1 = \tilde{b}_3^{d^2} = \left(\gamma \cdot \tilde{b}_1\tilde{b}_2\right)^{d^2} = \gamma^{d^2}\left(\tilde{b}_1\tilde{b}_2\right)^{d^2} = \gamma^{d^2} \cdot 1 = \gamma^{d^2}.$$

This implies that $\gamma^{d^2} = 1$, that is, $\gamma \in \mu_{d^2}$ and therefore

$$\tilde{b}_1\tilde{b}_2 = \gamma^{-1}\tilde{b}_3 \in \widetilde{B}.$$

This ends the proof.

THEOREM 7.17. *Let (A, e) be a symplectic pair. Suppose that $\bar{A} = A/\ker(e)$ is finite. Assume also that either* $\ker(e)$ *is divisible or A is finite. Let G be a theta group attached to (A, e).*

Then G is a Jordan group and its Jordan index equals $\sqrt{\#(\bar{A})}$.

PROOF. Assume that G sits in a short exact sequence (14). We can view $\mathbb{C}^*$ as a central subgroup of G. Let $\widetilde{\mathcal{A}}$ be a finite subgroup of G and $\widetilde{B}$ be a commutative subgroup of *maximum order* in $\widetilde{\mathcal{A}}$. Then $\widetilde{B}$ contains the intersection $\widetilde{\mathcal{A}} \cap \mathbb{C}^*$, and therefore the index $[\widetilde{\mathcal{A}} : \widetilde{B}]$ coincides with the index $[j(\widetilde{\mathcal{A}}) : j(\widetilde{B})]$. The commutativity of $\widetilde{B}$ means that $j(\widetilde{B})$ is an *isotropic* subgroup in $j(\widetilde{\mathcal{A}})$. This implies that

$$J_G \geqslant D_{A,e}.$$

Conversely, let $\mathcal{A}$ be a finite subgroup of A and B be an isotropic subgroup of maximal order in $\mathcal{A}$. By Lemma 7.16 there is a finite subgroup $\widetilde{\mathcal{A}}$ of G such that

$$j(\widetilde{\mathcal{A}}) = \widetilde{\mathcal{A}}.$$

Let $\widetilde{B}$ be the preimage of B in $\widetilde{\mathcal{A}}$. Then

$$j(\widetilde{B}) = B, \quad [\mathcal{A} : B] = [\widetilde{\mathcal{A}} : \widetilde{B}].$$

By Remark 7.14(ii), $\widetilde{B}$ is commutative because its image B is isotropic. The equality of indices implies that

$$J_G \leqslant D_{A,e},$$

which, combined with the previous opposite inequality, implies that $J_G = D_{A,e}$. Now the explicit formula for J_G follows from Remark 7.11.

8. Symplectic linear algebra

In this section we construct theta groups that arise from (not necessarily non-degenerate) alternating bilinear forms on integral lattices.

Definition 8.1. (i) An *admissible triple* is a triple (V, E, Π) that consists of a non-zero real vector space V of finite positive even dimension $2g$, an alternating $\mathbb{R}$-bilinear form

$$E\colon V \times V \to \mathbb{R}$$

on V, and a discrete lattice Π of rank $2g$ in V such that $E(\Pi, \Pi) \subset \mathbb{Z}$. Put

$$\Pi_E^{\perp} := \{v \in V \mid E(v, l) \in \mathbb{Z}\ \forall l \in \Pi\}.$$

By definition $\Pi_E^{\perp}$ is a closed real Lie subgroup of V that contains Π as a discrete subgroup.

(ii) A symplectic pair attached to the admissible triple (V, E, Π) is a pair $(K_{E,\Pi}, e_E)$ where $K_{E,\Pi} := \Pi_E^{\perp}/\Pi$ and the bilinear pairing e_E is defined as follows:

$$e_E\colon \Pi_E^{\perp}/\Pi \times \Pi_E^{\perp}/\Pi \to \mathbb{C}^*, \quad (v_1 + \Pi, v_2 + \Pi) \mapsto \exp(2\pi \mathbf{i} E(v_1, v_2)).$$

Definition 8.2. Recall that a subgroup C of a commutative group D is called *saturated* if it enjoys the following equivalent properties:

- there are no elements of finite order in the quotient D/C except 0;
- if x is an element of D such that there is a positive integer m with $mx \in C$, then $x \in C$.

Our goal is to find the isotropy index of $(K_{E,\Pi}, e_E)$. In order to do this, consider the *kernel* of E, that is, the subset

$$\ker(E) = \{v \in V \mid E(v, V) = \{0\}\} \subset V.$$

Clearly, $\ker(E)$ is a real even-dimensional (recall that E is alternating) vector subspace of V containing $\Pi_E^{\perp}$. Put

$$\Pi_0 := \Pi \cap \ker(E) \subset \ker(E).$$

Clearly, Π_0 is a *saturated* subgroup of Π. The integrality property of E implies that the natural homomorphism of real vector spaces

$$\Pi_0 \otimes \mathbb{R} \to \ker(E), \quad l_0 \otimes \lambda \mapsto \lambda \cdot l_0 \quad \forall\, l_0 \in \Pi_0,\ \lambda \in \mathbb{R}$$

is an isomorphism. In particular, the following conditions are equivalent:

(a) E is non-degenerate, that is, $\ker(E) = \{0\}$;

(b) $\Pi_0 = \{0\}$.

Consider several cases.

Case I. If $E \equiv 0$, then

$$\Pi_E^{\perp} = V, \quad K_{E,\Pi} = \Pi_E^{\perp}/\Pi = V/\Pi, \quad e_E \equiv 1,$$

$\ker(e_E) = K_{E,\Pi}$ is divisible, and $K_{E,\Pi}/\ker(e) = \{0\}$ is finite. By Remark 7.11 the isotropy defect $D_{K_{E,\Pi}, e_E}$ equals 1.

Case II. Suppose that E is a *non-degenerate* form. Let $\{s_1, \ldots, s_{2g}\}$ be any basis of the $\mathbb{Z}$-module Π. Clearly, it is also a basis of the $\mathbb{R}$-vector space V. Let

$$\widetilde{E} = \big(E(s_j, s_k)\big) \in \mathrm{Mat}_{2g}(\mathbb{Z})$$

be the $2g \times 2g$ *skew-symmetric* matrix of E with *integer* entries with respect to this basis. Let $\det(\widetilde{E})$ and $\mathrm{Pf}(\widetilde{E})$ be the *determinant* of $\widetilde{E}$ and the *Pfaffian* of $\widetilde{E}$, respectively. Then

$$\det(\widetilde{E}) \in \mathbb{Z}, \quad \mathrm{Pf}(\widetilde{E}) \in \mathbb{Z}; \qquad 0 \neq \det(\widetilde{E}) = \mathrm{Pf}(\widetilde{E})^2.$$

In particular, $\det(\widetilde{E})$ is a *positive integer*. Clearly, $\det(\widetilde{E})$ does not depend on the choice of a basis of Π, and therefore $|\mathrm{Pf}(\widetilde{E})|$ does not depend on this choice either. That is why we denote $\det(\widetilde{E})$ by $\det(E, \Pi)$ and $|\mathrm{Pf}(\widetilde{E})|$ by $|\mathrm{Pf}(E, \Pi)|$.

We claim that $\Pi_E^{\perp}/\Pi$ *is finite*, *the form*

$$e_E \colon \Pi_E^{\perp}/\Pi \times \Pi_E^{\perp}/\Pi \to \mathbb{C}^*$$

is non-degenerate, *and its isotropy defect is* $|\mathrm{Pf}(E, \Pi)|$.

Indeed, there is a basis $\{f_1, h_1, \ldots, f_g, h_g\}$ of Π such that

$$E(f_j, h_k) = -E(h_k, f_j) = 0 \quad \forall\, j \neq k \ (1 \leqslant j, k \leqslant g)$$

[37; Chap. XV, Exercise 17 on p. 598 (English ed.) or Chap. XIV, Exercise 4 on p. 426 (Russian ed.)]. Put

$$d_j = E(f_j, h_j) \in \mathbb{Z} \quad \forall j = 1, \ldots, g.$$

The non-degeneracy of E means that $d_j \neq 0$ for all j. Replacing h_j by $-h_j$ if necessary, we may and will assume that $d_j > 0$ for all j. If $\widetilde{E}$ is the matrix of E with respect to this basis, then the *Pfaffian* $\mathrm{Pf}(\widetilde{E})$ of $\widetilde{E}$ is $\pm \prod_{j=1}^{g} d_j$, and therefore

$$|\mathrm{Pf}(E, \Pi)| = \prod_{j=1}^{g} d_j.$$

We claim that

$$\Pi_E^{\perp} = \bigoplus_{j=1}^{g} \frac{1}{d_j}(\mathbb{Z} \cdot f_j \oplus \mathbb{Z} \cdot h_j). \tag{17}$$

Indeed, a vector

$$v = \left(\sum_{j=1}^{g} \lambda_j f_j\right) + \left(\sum_{j=1}^{g} \mu_j h_j\right) \quad \text{with all } \lambda_j, \mu_j \in \mathbb{R}$$

lies in $\Pi_E^{\perp}$ if and only if

$$\mathbb{Z} \ni E(f_j, v) = d_j \mu_j, \quad \mathbb{Z} \ni (h_j, v) = -d_j \lambda_j \quad \forall j,$$

which is obviously equivalent to (17).

It follows from (17) that

$$\Pi_E^{\perp}/\Pi = \bigoplus_{j=1}^{g} \frac{1}{d_j}(\mathbb{Z}\cdot f_j \oplus \mathbb{Z}\cdot h_j)/(\mathbb{Z}\cdot f_j \oplus \mathbb{Z}\cdot h_j). \tag{18}$$

Clearly, different summands of $\Pi_E^{\perp}/L$ are mutually orthogonal with respect to e_E while the restriction of e_E to each

$$\frac{1}{d_j}(\mathbb{Z}\cdot f_j \oplus \mathbb{Z}\cdot h_j)/(\mathbb{Z}\cdot f_j \oplus \mathbb{Z}\cdot h_j)$$

is isomorphic to $(\mathbf{S}_{d_j}, \mathbf{e}_{d_j})$. In particular, this restriction is a non-degenerate symplectic pair. This implies that the direct sum $(\Pi_E^{\perp}/\Pi, e_E)$ is also a non-degenerate symplectic pair. On the other hand, clearly,

$$\Pi_E^{\perp}/\Pi \cong \bigoplus_{j=1}^{g}\left(\frac{1}{d_j}\mathbb{Z}/\mathbb{Z}\right)^2.$$

Therefore,

$$\#(\Pi_E^{\perp}/\Pi) \cong \prod_{j=1}^{g} d_j^2, \qquad \sqrt{\#(\Pi_E^{\perp}/\Pi)} = \prod_{j=1}^{g} d_j = |\mathrm{Pf}(E,\Pi)|.$$

This implies that $(K_{E,\Pi}, e_E)$ is almost isotropic and its *isotropy defect* is $|\mathrm{Pf}(E,\Pi)|$.

Case IIbis. We keep the notation and assumptions of Case II. Consider the form nE, where n is a positive integer. Then

$$\Pi_{nE}^{\perp} = \frac{1}{n}\Pi_E^{\perp} = \bigoplus_{j=1}^{g} \frac{1}{nd_j}(\mathbb{Z}\cdot f_j \oplus \mathbb{Z}\cdot h_j),$$

$$\Pi_{nE}^{\perp}/\Pi \cong \bigoplus_{j=1}^{g}\left(\frac{1}{nd_j}\mathbb{Z}/\mathbb{Z}\right)^2,$$

$$\#(\Pi_{nE}^{\perp}/\Pi) = \prod_{j=1}^{g}(nd_j)^2, \qquad \sqrt{\#(\Pi_E^{\perp}/\Pi)} = n^g \prod_{j=1}^{g} d_j = n^g\cdot|\mathrm{Pf}(E,\Pi)|.$$

Hence the corresponding isotropy index

$$D_{K_{nE,\Pi}, e_{nE}} = n^g \cdot |\mathrm{Pf}(E,\Pi)|$$

for all positive integers n.

Case III. Now consider the case of degenerate non-zero E, that is, the case when

$$\{0\} \ne \Pi_0 \ne \Pi.$$

Clearly, Π_0 is a free abelian group of some positive even rank $2g_0 < 2g$. Since Π_0 is a saturated subgroup of Π, it is a *direct summand* of Π, that is, there is a (non-zero saturated) subgroup Π_1 of Π that is a free abelian group of rank $2g - 2g_0$ and such that

$$\Pi = \Pi_0 \oplus \Pi_1.$$

In other words, there is a basis $\{u_1, \dots, u_{2g_0}; v_1, \dots, v_{2g-2g_0}\}$ of the $\mathbb{Z}$-module Π such that $\{u_1, \dots, u_{2g_0}\}$ is a basis of Π_0 and $\{v_1, \dots, v_{2g-2g_0}\}$ is a basis of Π_1. Consider the real vector subspaces

$$V_0 := \sum_{j=1}^{2g_0} \mathbb{R}u_j \subset V, \qquad V_1 := \sum_{k=1}^{2g_1} \mathbb{R}v_k \subset V.$$

Clearly,

$$V = V_0 \oplus V_1, \qquad \Pi_0 = V_0 \cap \Pi, \quad \Pi_1 = V_1 \cap \Pi;$$

in addition, $V_0 = \ker(E)$, the subspaces V_0 and V_1 are mutually orthogonal with respect to E and the restriction of E to V_1,

$$E_1 \colon V_1 \times V_1 \to \mathbb{R}, \qquad u, v \mapsto E(u, v),$$

is a *non-degenerate* alternating bilinear form. It is also clear that

$$E_1(\Pi_1, \Pi_1) = E(\Pi_1, \Pi_1) \subset E(\Pi, \Pi) \subset \mathbb{Z}.$$

On the other hand, the restriction of E to V_0, which we denote by E_0, is identically 0. This implies that (as the symplectic pair)

$$(K_{E,\Pi}, e_E) = (K_{E_0,\Pi_0}, e_{E_0}) \oplus (K_{E_1,\Pi_1}, e_{E_1}).$$

By Case I as applied to (V_0, E_0, Π_0), the group $K_{E_0,\Pi_0} = V_0/\Pi_0$ is *divisible* as a quotient of a complex vector space, and $e_{E_0} \equiv 1$. By Case II as applied to (V_1, E_1, Π_1), the group K_{E_1,Π_1} is *finite* of order $|\mathrm{Pf}(E, \Pi)|^2$ and the pairing

$$e_{E_1} \colon K_{E_1,\Pi_1} \times K_{E_1,\Pi_1} \to \mathbb{C}^*$$

is *non-degenerate*. Hence $\ker(e_E) = K_{E_0,\Pi_0}$, and therefore $\ker(e_E)$ is divisible and

$$K_{E,\Pi}/\ker(e_E) = K_{E_1,\Pi_1}$$

is a finite group. This implies that $(K_{E,\Pi}, e_E)$ is almost isotropic and its isotropy defect is

$$D_{K_{E,\Pi}, e_E} = \sqrt{\#(K_{E,\Pi}/\ker(e_E))} = \sqrt{\#(K_{E_1,\Pi_1})} = |\mathrm{Pf}(E_1, \Pi_1)| \tag{19}$$

by Theorem 7.17.

Case IIIbis. We keep the notation and assumptions of Case III. Let

$$M \colon V \times V \to \mathbb{R}$$

be an alternating bilinear form that enjoys the following properties:

(a) $M(\Pi, \Pi) \subset \mathbb{Z}$;

(b) $\ker(E) \subset \ker(M)$.

If n is an integer, then we write $\mathbf{M}(n)$ for the alternating bilinear form $nE + M$ on V. Clearly,

$$\mathbf{M}(n)(\Pi, \Pi) \subset nE(\Pi, \Pi) + M(\Pi, \Pi) \subset n\mathbb{Z} + \mathbb{Z} = \mathbb{Z}.$$

Lemma 8.3. *There exists a degree $g - g_0$ polynomial $\mathcal{P}(t) \in \mathbb{Z}[t]$ with integer coefficients and leading coefficient $|\mathrm{Pf}(E_1, \Pi_1)|$ that enjoys the following property: For all but finitely many positive integers n the symplectic pair $(K_{\mathbf{M}(n),\Pi}, e_{\mathbf{M}(n)})$ is almost isotropic with isotropy defect*

$$D_{K_{\mathbf{M}(n),\Pi}, e_{\mathbf{M}(n)}} = \mathcal{P}(n). \tag{20}$$

Proof. Indeed, let $M_1 \colon V_1 \times V_1 \to \mathbb{R}$ be the restriction of M to V_1. Let $\widetilde{E}_1$ and $\widetilde{M}_1$ be the matrices of E_1 and M_1 with respect to the basis $\{f_1, \dots, f_{2g-2g_0}\}$ of Π_1. The non-degeneracy of E_1 implies that $\det(\widetilde{E}_1) \neq 0$, and therefore the determinant

$$\det(n\widetilde{E}_1 + \widetilde{M}_1) = \det(\widetilde{E}_1)\det(n\,\mathrm{I}_{2g-2g_0} + \widetilde{E}_1^{-1}\widetilde{M}_1)$$

does *not* vanish for all but finitely many integers n. (Here and in what follows I_{2g-2g_0} is the identity square matrix of size $2g - 2g_0$.) Taking into account that $n\widetilde{E}_1 + \widetilde{M}_1$ is the matrix of the restriction of $nE + M = \mathbf{M}(n)$, we obtain that for all but finitely many integers n

$$\ker(\mathbf{M}(n)) = \ker(nE + M) = \ker(E) = V_0. \tag{21}$$

In what follows we assume that n is any integer that enjoys property (21) (this assumption excludes only finitely many integers n). Now we can apply the results of Case III to $\mathbf{M}(n) = nE + M$ (instead of E) and get that $(K_{\mathbf{M}(n),\Pi}, e_{\mathbf{M}(n)})$ is almost isotropic and its isotropy defect is

$$\begin{aligned}|\mathrm{Pf}(nE_1 + M_1, \Pi_1)| &= \sqrt{\det(nE_1 + M_1)} = \sqrt{\det(\widetilde{E}_1)\det\bigl(n\,\mathrm{I}_{2g-2g_0} + \widetilde{E}_1^{-1}\widetilde{M}_1\bigr)} \\ &= |\mathrm{Pf}(E_1, \Pi_1)|\sqrt{\det\bigl(n\,\mathrm{I}_{2g-2g_0} + \widetilde{E}_1^{-1}\widetilde{M}_1\bigr)}.\end{aligned}$$

Clearly, there is a polynomial $\mathcal{Q}(t) \in \mathbb{Z}[t]$ with integer coefficients such that for *all* n under consideration

$$\mathcal{Q}(n) = \mathrm{Pf}(n\widetilde{E}_1 + \widetilde{M}_1).$$

This implies that

$$\mathcal{Q}(n)^2 = \det(n\widetilde{E}_1 + \widetilde{M}_1) = \det(\widetilde{E}_1)\det\bigl(n\,\mathrm{I}_{2g-2g_0} + \widetilde{E}_1^{-1}\widetilde{M}_1\bigr).$$

It is also clear that there exists a *monic* degree $2g - 2g_0$ polynomial $\mathcal{R}(t) \in \mathbb{Q}[t]$ with rational coefficients such that for all our n

$$\mathcal{R}(n) = \det\bigl(n\,\mathrm{I}_{2g-2g_0} + \widetilde{E}_1^{-1}\widetilde{M}_1\bigr).$$

This implies that

$$\mathcal{Q}(n)^2 = \det(\widetilde{E}_1)\mathcal{R}(n) = |\mathrm{Pf}(E_1, \Pi_1)|^2\mathcal{R}(n).$$

Since $\mathcal{R}(t)$ is monic of degree $2g - 2g_0$, we have

$$\deg(\mathcal{Q}) = g - g_0$$

and the leading coefficient of $\mathcal{Q}(t)$ is $\pm|\mathrm{Pf}(E_1, \Pi_1)|$.

Let $\mathcal{P}(t)$ be the polynomial with positive leading coefficient that coincides either with $\mathcal{Q}(t)$ or with $-\mathcal{Q}(t)$. Then $\mathcal{P}(t)$ is a degree $g-g_0$ polynomial with integer coefficients and leading coefficient $|\mathrm{Pf}(E_1,\Pi_1)|$ such that

$$\mathcal{P}(n)=\pm\,\mathrm{Pf}(n\widetilde{E}_1+\widetilde{M}_1).$$

Since the leading coefficient of $\mathcal{P}(t)$ is positive, we see that $\mathcal{P}(n)$ is positive for all but finitely many positive integers n. Hence

$$\mathcal{P}(n)=|\mathrm{Pf}(n\widetilde{E}_1+\widetilde{M}_1)|=|\mathrm{Pf}(nE_1+M_1,\Pi_1)|$$

for all such n. This completes the proof.

THEOREM 8.4. *Let g be a positive integer, V be a $2g$-dimensional real vector space, and (V,E,Π) and (V,M,Π) be admissible triples such that*

$$E\not\equiv 0 \quad \textit{and} \quad \ker(E)\subset\ker(M).$$

If n is an integer, then we write $\mathbf{M}(n)$ for the alternating bilinear form $nE+M$ on V.

Let $\mathcal{G}$ be a group that enjoys the following properties: there are infinitely many positive integers n such that $\mathcal{G}$ contains a subgroup G_n that is a theta group attached to $(K_{\mathbf{M}(n),\Pi},e_{\mathbf{M}(n)})$.

Then $\mathcal{G}$ is not Jordan.

PROOF. It suffices to check that the Jordan index of G_n tends to infinity as n tends to infinity. But this assertion follows from the results of Cases II, III, IIIbis of this section combined with Theorem 7.17.

9. Line bundles over tori and theta groups

In this section we use results from the previous two sections in order to compute the Jordan index of certain automorphism groups of holomorphic line bundles on complex tori.

Let V be a complex vector space of finite positive dimension g, Π be a discrete lattice of rank $2g$ in V, and

$$H\colon V\times V\to\mathbb{C}$$

be an Hermitian form on V such that its imaginary part

$$E\colon V\times V\to\mathbb{R},\quad (v_1,v_2)\mapsto \mathrm{Im}(H(v_1,v_2)),$$

satisfies

$$E(\Pi,\Pi)\subset\mathbb{Z}.$$

One can view V as the $2g$-dimensional real vector space. Then E becomes an alternating $\mathbb{R}$-bilinear form on V such that

$$E(\mathbf{i}v_1,\mathbf{i}v_2)=E(v_1,v_2)\quad\forall v_1,v_2\in V.$$

In addition,

$$H(v_1, v_2) = E(\mathbf{i}v_1, v_2) + \mathbf{i}E(v_1, v_2) \quad \forall v_1, v_2 \in V$$

(see [12; Lemma 2.1.7]). This implies that H and E have the same kernels, that is,

$$\ker(H) := \{w \in V \mid H(w, V) = 0\} = \{w \in V \mid E(w, V) = 0\} =: \ker(E).$$

Definition 9.1 (see [11], [33]). A pair (H, α) is called *Appell–Humbert data* (A.-H. data) on (V, Π) if H, E, Π are as above and α is a map ("semicharacter")

$$\alpha\colon \Pi \to \mathbf{U}(1) = \{z \in \mathbb{C}\colon |z| = 1\} \subset \mathbb{C}^*$$

such that

$$\alpha(l_1 + l_2) = (-1)^{E(l_1, l_2)}\alpha(l_1)\alpha(l_2) \quad \forall l_1, l_2 \in \Pi. \tag{22}$$

In particular, if $l_1 = l_2 = 0$, then $\alpha(0) = \alpha(0)^2$, that is,

$$\alpha(0) = 1.$$

Notice that a classical theorem of Appell and Humbert ([33; Theorem 1.5], [11; Theorem 21.1]) classifies holomorphic line bundles on the complex torus V/Π in terms of A.-H. data.

The construction in section 8 gives us the symplectic pair $(K_{E,\Pi}, e_E)$. The aim of this section is to construct a theta group $\mathfrak{G}(H, \alpha)$ attached to this pair that corresponds to any A.-H. data (H, α). We define $\widetilde{\mathfrak{G}}(H, V)$ to be a certain group of biholomorphic automorphisms of $\mathcal{L}(H, \alpha)$. Here $\mathcal{L}(H, \alpha)$ is the *total body* of the holomorphic line bundle $\mathcal{L}(H, \alpha)$ over V/Π that corresponds to A.-H. data (H, α).

First, we start with a certain theta group $\widetilde{\mathfrak{G}}(H, V)$ attached to the symplectic pair $(V, \tilde{e}_E)$ where

$$\tilde{e}_E\colon V \times V \to \mathbb{C}^*, \quad (v_1, v_2) \mapsto \exp(2\pi\mathbf{i}E(v_2, v_1)).$$

We define $\widetilde{\mathfrak{G}}(H, V)$ as a certain group of holomorphic automorphisms of

$$V_{\mathbb{L}} := V \times \mathbb{L}$$

where $\mathbb{L}$ is a one-dimensional $\mathbb{C}$-vector space. Namely, $\widetilde{\mathfrak{G}}(H, V)$ consists of the automorphisms $\mathcal{B}_{H,u,\lambda}$ indexed by $u \in V$, $\lambda \in \mathbb{C}^*$ that are defined as follows:

$$\mathcal{B}_{H,u,\lambda}\colon (v, c) \mapsto \bigl(v + u, \lambda \exp(\pi H(v, u)c)\bigr) \quad \forall\, v \in V,\ c \in \mathbb{L}.$$

One may easily check (see [87; § 2.1]) that

$$\mathcal{B}_{H,u_1,\lambda_1} \circ \mathcal{B}_{H,u_2,\lambda_2} = \mathcal{B}_{H,u_1+u_2,\lambda_1\lambda_2\mu} \quad \text{where} \quad \mu = \exp(\pi H(u_2, u_1)) \tag{23}$$

and the inverse is

$$\mathcal{B}_{H,u,\lambda}^{-1} = \mathcal{B}_{H,-u,\nu/\lambda} \quad \text{where} \quad \nu = \exp(-\pi H(u, u)). \tag{24}$$

This implies that $\widetilde{\mathfrak{G}}(H, V)$ is indeed a subgroup of the group of biholomorphic automorphisms of $V_{\mathbb{L}}$. (Our $\mathfrak{G}(H, \alpha)$ will be defined as a subquotient of $\widetilde{\mathfrak{G}}(H, V)$.)

Notice that for all $\lambda \in \mathbb{C}^*$ the automorphism $\mathcal{B}_{H,0,\lambda}$ sends every (u,c) to $(u,\lambda c)$. This implies that the map

$$\mathrm{mult}\colon \mathbb{C}^* \to \widetilde{\mathfrak{G}}(H,V), \quad \lambda \mapsto \mathcal{B}_{H,0,\lambda},$$

is an injective group homomorphism, whose image lies in the center of $\widetilde{\mathfrak{G}}(H,V)$. This allows us to include $\widetilde{\mathfrak{G}}(H,V)$ in a short exact sequence of groups

$$1 \to \mathbb{C}^* \xrightarrow{\mathrm{mult}} \widetilde{\mathfrak{G}}(H,V) \xrightarrow{\tilde{j}} V \to 0$$

where $\tilde{j}$ sends $\mathcal{B}_{H,u,\lambda}$ to u. It follows from (23) and (24) (see also [87; § 2.1]) that

$$\begin{aligned}\mathcal{B}_{H,u_1,\lambda_1} \circ \mathcal{B}_{H,u_2,\lambda_2} \circ \mathcal{B}^{-1}_{H,u_1,\lambda_1} \circ \mathcal{B}^{-1}_{H,u_2,\lambda_2} \\ = \mathrm{mult}(\exp(2\pi\mathbf{i}E(u_2,u_1))) = \mathrm{mult}(\tilde{e}_E(u_1,u_2)).\end{aligned} \tag{25}$$

This implies that $\widetilde{\mathfrak{G}}(H,V)$ is a theta group attached to the symplectic pair $(V,\tilde{e}_E)$.

Consider the following subgroups of $\widetilde{\mathfrak{G}}(H,V)$:

$$\widetilde{\mathfrak{G}}(H,\Pi) = \tilde{j}^{-1}(\Pi) = \{\mathcal{B}_{H,u,\lambda} \mid \lambda \in \mathbb{C}^*, u \in \Pi\}; \tag{26}$$

$$\widetilde{\mathfrak{G}}(H,\Pi_E^{\perp}) = \tilde{j}^{-1}(\Pi_E^{\perp}) = \{\mathcal{B}_{H,u,\lambda} \mid \lambda \in \mathbb{C}^*, u \in \Pi_E^{\perp}\}. \tag{27}$$

By Remark 7.15, $\widetilde{\mathfrak{G}}(H,\Pi)$ and $\widetilde{\mathfrak{G}}(H,\Pi_E^{\perp})$ are theta groups attached to the symplectic pairs $(\Pi, \tilde{e}|_{\Pi})$ and $(\Pi_E^{\perp}, \tilde{e}|_{\Pi_E^{\perp}})$, respectively. Since $\Pi \subset \Pi_E^{\perp}$, the group $\widetilde{\mathfrak{G}}(H,\Pi)$ is a subgroup of $\widetilde{\mathfrak{G}}(H,\Pi_E^{\perp})$. It follows from (25) that $\widetilde{\mathfrak{G}}(H,\Pi)$ is actually a *central* subgroup of $\widetilde{\mathfrak{G}}(H,\Pi_E^{\perp})$, because

$$E(\Pi,\Pi_E^{\perp}) = \{0\}.$$

We define $\mathfrak{G}(H,\alpha)$ as the quotient of $\widetilde{\mathfrak{G}}(H,\Pi_E^{\perp})$ by a certain central subgroup that depends on the "semicharacter" α. In order to define this subgroup, let us consider the *discrete free* action of the group Π on $V_{\mathbb{L}}$ by holomorphic automorphisms defined as follows. An element l of Π acts as

$$\mathcal{A}_{H,\alpha,l}\colon V_{\mathbb{L}} \to V_{\mathbb{L}}, \quad (v,c) \mapsto \left(v+l, c\alpha(l)\exp\left(\pi H(v,l) + \frac{1}{2}\pi H(l,l)\right)\right) \tag{28}$$
$$\forall\, v \in V,\ c \in \mathbb{L},$$

that is,

$$\mathcal{A}_{H,\alpha,l} = \mathrm{mult}(\alpha(l))\mathcal{B}_{H,l,1} \in \widetilde{\mathfrak{G}}(H,\Pi). \tag{29}$$

Direct calculations based on (22) show that

$$\mathcal{A}_{H,\alpha,l_1}\mathcal{A}_{H,\alpha,l_2} = \mathcal{A}_{H,\alpha,l_1+l_2} \quad \forall l_1,l_2 \in \Pi,$$

that is,

$$\mathbf{A}^{\Pi}\colon \Pi \to \widetilde{\mathfrak{G}}(H,\Pi), \quad l \mapsto \mathcal{A}_{H,\alpha,l},$$

is an *injective* group homomorphism, whose image we denote by

$$\widetilde{\Pi} = \widetilde{\Pi}(H,\alpha) := \mathbf{A}^{\Pi}(\Pi) \subset \widetilde{\mathfrak{G}}(H,\Pi) \subset \widetilde{\mathfrak{G}}(H,\Pi_E^{\perp}).$$

Notice that $\widetilde{\Pi}$ meets $\mathrm{mult}(\mathbb{C}^*)$ precisely at the identity element of $\widetilde{\mathfrak{G}}(H, \Pi_E^\perp)$. Notice that the quotient $V_{\mathbb{L}}/\widetilde{\Pi}(H,\alpha)$ is precisely the total body $\mathcal{L}(H,\alpha)$ of the holomorphic vector bundle $\mathcal{L}(H,\alpha)$ over V/Π attached to the A.-H. data (H,α) where the structure map

$$p\colon \mathcal{L}(H,\alpha) = V_{\mathbb{L}}/\widetilde{\Pi}(H,\alpha) \to V/\Pi$$

is induced by the projection map

$$V_{\mathbb{L}} = V \times \mathbb{L} \to V$$

[12; Chap. 2, § 2.2, p. 30]. Put

$$\mathfrak{G}(H,\alpha) := \widetilde{\mathfrak{G}}(H, \Pi_E^\perp)/\widetilde{\Pi}(H,\alpha). \tag{30}$$

The faithful action of $\widetilde{\mathfrak{G}}(H, \Pi_E^\perp)$ on $V_{\mathbb{L}}$ induces a *faithful* action of $\mathfrak{G}(H,\alpha)$ on $\mathcal{L}(H,\alpha)$. Under this action each coset

$$\mathcal{B}_{H,u,\lambda}\widetilde{\Pi} \in \widetilde{\mathfrak{G}}(H, \Pi_E^\perp)/\widetilde{\Pi}(H,\alpha) = \mathfrak{G}(H,\alpha)$$

maps $\mathbb{C}$-*linearly* and isomorphically the fibre of p over $v + \Pi \in V/\Pi$ to the fibre over $(v+u)\Pi \in V/\Pi$ for any pair

$$u + \Pi \in \Pi_E^\perp/\Pi \subset V/\Pi, \quad \text{and} \quad v + \Pi \in V/\Pi, \quad \text{and} \quad \lambda \in \mathbb{C}^*.$$

In particular, $\mathrm{mult}(\lambda)\widetilde{\Pi}$ acts as the automorphism $[\lambda]$ that leaves invariant each fibre of $p\colon \mathcal{L}(H,\alpha) \to V/\Pi$ and acts on this fibre (which is a one-dimensional $\mathbb{C}$-vector space) as multiplication by λ (for all $\lambda \in \mathbb{C}^*$). Clearly, each $[\lambda]$ lies in the *centre* of $\mathfrak{G}(H,\alpha)$.

Lemma 9.2. *The group $\mathfrak{G}(H,\alpha)$ is a theta group attached to the symplectic pair $(K_{E,\Pi}, e_E)$.*

Proof. Clearly,

$$[\mathrm{mult}]\colon \mathbb{C}^* \to \mathfrak{G}(H,\alpha), \quad \lambda \mapsto [\lambda],$$

is an *injective* group homomorphism, whose image $[\mathrm{mult}](\mathbb{C}^*)$ is a *central* subgroup of $\mathfrak{G}(H,\alpha)$. On the other hand, $\tilde{j}$ induces the surjective group homomorphism

$$j\colon \mathfrak{G}(H,\alpha) = \widetilde{\mathfrak{G}}(H, \Pi_E^\perp)/\widetilde{\Pi} \twoheadrightarrow \Pi_E^\perp/\Pi = K_{E,\Pi},$$
$$\mathcal{B}_{H,u,\lambda}\widetilde{\Pi} \mapsto u + \Pi \in \Pi_E^\perp/\Pi.$$

Clearly, the kernel of j consists of all $\mathcal{B}_{H,0,\lambda}\widetilde{\Pi} = [\mathrm{mult}](\lambda)$, that is, it coincides with $[\mathrm{mult}](\mathbb{C}^*)$. Hence $\mathfrak{G}(H,\alpha)$ sits in the short exact sequence

$$1 \to \mathbb{C}^* \xrightarrow{[\mathrm{mult}]} \mathfrak{G}(H,\alpha) \xrightarrow{j} \Pi_E^\perp/\Pi \to 0.$$

It follows from (25) that $\mathfrak{G}(H,\alpha)$ is a theta group attached to the symplectic pair $(K_{E,\Pi}, e_E)$.

Remark 9.3. It is well known [12; Lemma 2.2.1] that if (H_1,α_1) and (H_2,α_2) are A.-H. data on (V,Π), then the pair $(H_1+H_2, \alpha_1\alpha_2)$ also is A.-H. data on (V,Π) and the holomorphic vector bundles $\mathcal{L}(H_1+H_2,\alpha_1\alpha_2)$ and $\mathcal{L}(H_1,\alpha_1) \otimes \mathcal{L}(H_2,\alpha_2)$ are canonically isomorphic.

10. $\mathbb{P}^1$-bundles bimeromorphic to the direct product

In this section we prove the non-Jordanness of the groups of bimeromorphic self-maps of certain $\mathbb{P}^1$-bundles over complex tori of positive algebraic dimension.

Let V be a complex vector space of finite positive dimension g, Π be a discrete lattice of rank $2g$ in V, and $T = V/\Pi$ be the corresponding complex torus. Recall that $\mathbf{1}_T$ stands for the trivial holomorphic line bundle $T \times \mathbb{C}$ over T. If x is point of T, then we write $\mathcal{L}_x$ for the fibre of a holomorphic vector bundle $\mathcal{L}$ over T, which is a one-dimensional complex vector space. We write $\bar{\mathcal{L}}$ for the projectivization $\mathbb{P}(\mathcal{E})$ of the two-dimensional holomorphic vector bundle $\mathcal{E} = \mathcal{L} \oplus \mathbf{1}_T$. The fibre $\mathcal{E}_x$ of $\mathcal{E}$ over x is the set of pairs (s_x, c), where $s_x \in \mathcal{L}_x$, $c \in \mathbb{C}$ and the fibre $\bar{\mathcal{L}}_x$ of $\bar{\mathcal{L}}$ over x is the set of equivalence classes of $(s_x : c)$, where either $s_x \neq 0$ or $c \neq 0$ and the equivalence class of $(s_x : c)$ is the set of all

$$(\mu s_x : \mu c), \quad \mu \in \mathbb{C}^*.$$

LEMMA 10.1. *Suppose that $\mathcal{L} = \mathcal{L}(H, \alpha)$, where (H, α) is A.-H. data. Then there is a natural group embedding*

$$\mathfrak{G}(H, \alpha) \hookrightarrow \mathrm{Aut}(\overline{\mathcal{L}(H, \alpha)}).$$

PROOF. First, let us define the group embedding

$$\mathfrak{G}(H, \alpha) \hookrightarrow \mathrm{Aut}(\mathcal{L}(H, \alpha) \oplus \mathbf{1}_T) \tag{31}$$

by the formula

$$\begin{gathered} g \colon (s_x, (x, c)) \mapsto (g(s_x), (x + j(g), c)) \\ \forall\, g \in \mathfrak{G}(H, \alpha),\ x \in V/\Pi = T,\ c \in \mathbb{C},\ s_x \in \mathcal{L}_x \subset \mathcal{L}. \end{gathered} \tag{32}$$

In particular, g induces an isomorphism of two-dimensional complex vector spaces between the fibres of $\mathcal{L}(H, \alpha) \oplus \mathbf{1}_T$ over x and over $x + j(g)$. Since $\mathfrak{G}(H, \alpha) \to \mathrm{Aut}(\mathcal{L}(H, \alpha))$ is a group *embedding*, we conclude that if $j(g) = 0$, then g_x is multiplication by a scalar if and only if g is the identity element of $\mathfrak{G}(H, \alpha)$. This implies that (31) and (32) induce a group embedding

$$\mathfrak{G}(H, \alpha) \hookrightarrow \mathrm{Aut}(\mathbb{P}(\mathcal{L}(H, \alpha) \oplus \mathbf{1}_T)) = \mathrm{Aut}(\overline{\mathcal{L}(H, \alpha)}) \tag{33}$$

such that each $g \in \mathfrak{G}(H, \alpha)$ sends every $(s_x : c) \in \mathcal{L}(H, \alpha)_x$ to $(g(s_x) : c) \in \mathcal{L}(H, \alpha)_{x+j(g)}$. This completes the proof.

Let $\mathcal{L}$ be a holomorphic line bundle over the complex torus $T = V/\Pi$. Then $\mathcal{L} \cong \mathcal{L}(H, \alpha)$ for certain (actually, unique) A.-H. data $H, \alpha)$ on (V, Π) [33; Theorem 1.5]. Let us denote by $\mathfrak{G}(\mathcal{L})$ the group $\mathfrak{G}(H, \alpha)$. By Lemma 10.1 there exists a *group embedding*

$$\mathfrak{G}(\mathcal{L}) \hookrightarrow \mathrm{Aut}(\bar{\mathcal{L}}). \tag{34}$$

LEMMA 10.2. *Let $\mathcal{L}$ and $\mathcal{N}$ be holomorphic line bundles over $T = V/\Pi$. Assume that $\mathcal{L}$ admits a non-zero holomorphic section. Then the compact complex manifolds $\overline{\mathcal{N}}$ and $\overline{\mathcal{L}^n \otimes \mathcal{N}}$ are bimeromorphic for all positive integers n. In particular, for all such n there is a* group embedding

$$\mathfrak{G}(\mathcal{L}^n \otimes \mathcal{N}) \hookrightarrow \mathrm{Bim}(\overline{\mathcal{N}}). \tag{35}$$

PROOF. Let t be a non-zero section of $\mathcal{L}$. Then t^n is a non-zero section of $\mathcal{L}^n$. So it suffices to prove the lemma for $n = 1$, that is, to prove that $\bar{\mathcal{L}}$ and $\overline{\mathcal{L} \otimes \mathcal{N}}$ are *bimeromorphic*.

The holomorphic $\mathbb{C}$-linear map of rank 2 vector bundles

$$\begin{gathered} \mathcal{N} \oplus \mathbf{1}_T \to (\mathcal{L} \otimes \mathcal{N}) \oplus \mathbf{1}_T, \\ (s_x; (x, c)) \mapsto (s_x \otimes t(x); (x, c)) \quad \forall\, x \in T,\ s_x \in \mathcal{N}_x,\ c \in \mathbb{C}, \end{gathered}$$

induces a bimeromorphic isomorphism of their projectivizations $\overline{\mathcal{N}}$ and $\overline{\mathcal{L} \otimes \mathcal{N}}$. Hence the groups $\mathrm{Bim}(\overline{\mathcal{N}})$ and $\mathrm{Bim}(\overline{\mathcal{L} \otimes \mathcal{N}})$ are isomorphic. Now the second assertion of our lemma follows from Lemma 10.1.

COROLLARY 10.3. *We keep the notation and assumptions of Lemma 10.2. In particular, $\mathcal{L}$ is isomorphic to $\mathcal{L}(H, \alpha)$ and admits a non-zero holomorphic section.*

Suppose that $\mathcal{N}$ is isomorphic to $\mathcal{L}(H_0, \beta)$, where the kernel $\ker(H_0)$ of the Hermitian form H_0 contains the kernel $\ker(H)$ of the Hermitian form H.

Then the group $\mathrm{Bim}(\overline{\mathcal{N}})$ is not Jordan.

PROOF. Consider the alternating $\mathbb{R}$-bilinear forms $E := \mathrm{Im}(H)$ and $M := \mathrm{Im}(H_0)$ on V. We have

$$\ker(E) = \ker(H) \subset \ker(H_0) = \ker(M),$$

and therefore $\ker(E) \subset \ker(M)$. Notice also that the alternating form $\mathbf{M}(n) = nE + M$ is the imaginary part of the Hermitian form $nH + H_0$ for all positive integers n; in addition, obviously, the holomorphic line bundle

$$\mathcal{L}^n \otimes \mathcal{N} \cong \mathcal{L}(H, \alpha)^n \otimes \mathcal{L}(H_0, \beta) = \mathcal{L}(nH + H_0, \alpha\beta^n) = \mathcal{L}(\mathbf{M}(n), \alpha\beta^n).$$

In light of Lemma 10.2 there is a group embedding

$$\mathfrak{G}(nH + H_0, \alpha\beta^n) \hookrightarrow \mathrm{Bim}(\overline{\mathcal{N}}).$$

On the other hand, applying Lemma 9.2 to $(nH + H_0, \alpha\beta^n)$ (instead of (H, α)), we conclude that $\mathfrak{G}(nH + H_0, \alpha\beta^n)$ is a *theta group* attached to the *symplectic pair* $(K_{\mathbf{M}(n),\Pi}, e_{\mathbf{M}(n)})$. Now the desired result follows from Theorem 8.4.

DEFINITION 10.4. Let $T = V/\Gamma$ be a complex torus. We write T_a for its algebraic model, which is also a complex torus (even an abelian variety) provided with a surjective holomorphic homomorphism of complex tori

$$\pi_a \colon T \twoheadrightarrow T_a$$

with connected kernel (actually, all the fibres of π_a are connected) [11; Chap. 2, § 6]. We write $\dim_a(T)$ for $\dim(T_a)$ and call it the *algebraic dimension* of T.

Clearly,

$$\dim(T_a) \leqslant \dim(T);$$

equality holds if and only if $T = T_a$, that is, T is an *abelian variety*.

THEOREM 10.5 (Theorem 1.7 of [87]). *Suppose that a complex torus $T = V/\Pi$ has a positive algebraic dimension. Then $\mathrm{Bim}(T \times \mathbb{P}^1)$ is not Jordan.*

PROOF. Take $\mathcal{N} = \mathbf{1}_T$. Then $\overline{\mathcal{N}} = T\times\mathbb{P}^1$. On the other hand $\mathcal{N} = \mathbf{1}_T \cong \mathcal{L}(\mathbf{0}, \mathbf{1})$, where $\mathbf{0}$ is the *zero* Hermitian form on V and

$$\mathbf{1}_\Pi \colon \Pi \to \{1\} \subset \mathbf{U}(1) \subset \mathbb{C}^*$$

is the constant semicharacter (actually, character) of Π that identically equals 1. Clearly,

$$\ker(\mathbf{0}) = V.$$

Since $\dim_a(T) > 0$, the algebraic model T_a is a *positive-dimensional* abelian variety. Then T_a admits an *ample* holomorphic line bundle $\mathcal{L}_a$ with a non-zero section. Since $\psi\colon T \to T_a$ is surjective, the inverse image $\mathcal{L} = \psi^*\mathcal{L}_a$ is a holomorphic line bundle on T that also admits a non-zero section. We have $\mathcal{L} \cong \mathcal{L}(H, \alpha)$ for some A.-H. data (H, α). Obviously,

$$\ker(H) \subset V = \ker(\mathbf{0}).$$

Therefore, we can apply Corollary 10.3 and obtain that the group $\mathrm{Bim}(\overline{\mathcal{N}})$ is *not* Jordan. It remains to recall that $\overline{\mathcal{N}} = T \times \mathbb{P}^1$.

The following assertion is a generalization of Theorem 10.5.

THEOREM 10.6 (special case of Theorem 1.8 in [87]). *Let $\psi\colon T \to A$ be a surjective holomorphic group homomorphism from a complex torus $T = V/\Pi$ to a positive-dimensional complex abelian variety A. Let $\mathcal{M}$ be a holomorphic line bundle over A and $\mathcal{F}$ be a holomorphic line bundle over T that is isomorphic to the inverse image $\psi^*\mathcal{M}$.*

Then the group $\mathrm{Bim}(\overline{\mathcal{F}})$ *is not Jordan.*

PROOF. A positive-dimensional complex abelian variety A is a complex torus $A = W/\Gamma$ (where W is a complex vector space of finite positive dimension m and Γ is a discrete lattice of rank $2m$ in W) that admits a *polarization*, that is, a *positive* (and therefore *non-degenerate*) Hermitian form

$$\mathbf{H}_A \colon W \times W \to \mathbb{C},$$

whose imaginary part

$$\mathbf{E}_A \colon W \times W \to \mathbb{R}, \quad (w_1, w_2) \mapsto \mathrm{Im}(\mathbf{H}_A(w_1, w_2)),$$

satisfies the condition

$$\mathbf{E}_A(\Gamma, \Gamma) \subset \mathbb{Z}.$$

Replacing $\mathbf{H}_A$ by $2\mathbf{H}_A$ if necessary, we may and will assume that

$$\mathbf{E}_A(\Gamma, \Gamma) \subset 2 \cdot \mathbb{Z}.$$

Then obviously $(\mathbf{H}_A, \mathbf{1}_\Gamma)$ is A.-H. data on (W, Γ). The positiveness of $\mathbf{H}_A$ implies that the corresponding holomorphic line bundle $\mathcal{L}(\mathbf{H}_A, \mathbf{1})$ over A has a *non-zero* holomorphic section (the corresponding theta function) (see [33; Theorem 2.1]).

It follows from [11; Lemma 2.3.4 on p. 33] that every surjective holomorphic homomorphism $\psi\colon T \to A$ is induced by some surjective $\mathbb{C}$-linear map $\bar{\psi}\colon V \to W$ in the sense that

$$\bar{\psi}(\Pi) \subset \Gamma; \qquad \psi(v + \Pi) = \bar{\psi}(v) + \Gamma \in W/\Gamma = A \quad \forall\, v + \Pi \in V/\Pi = T.$$

The surjectiveness of ψ implies that the induced holomorphic line bundle $\mathcal{L} = \psi^*\mathcal{L}(\mathbf{H}_A, \mathbf{1}_\Gamma)$ over T also has a *non-zero* holomorphic section.

Let (H_A, β) be A.-H. data on (W, Γ) and $\mathcal{L}(H_A, \beta)$ be the corresponding holomorphic line bundle over $A = W/\Gamma$. Then the inverse image $\psi^*\mathcal{L}(H_A, \beta)$ is isomorphic to $\mathcal{L}(H_A \circ \bar{\psi}, \beta \circ \bar{\psi})$ where the A.-H. data $(H_A \circ \bar{\psi}, \beta \circ \bar{\psi})$ for (V, Γ) are as follows (see [33; Lemma 2.3.4]):

$$\begin{gathered} H_A \circ \bar{\psi}\colon V \times V \to \mathbb{C}, \quad (v_1, v_2) \mapsto H_A(\bar{\psi}v_1, \bar{\psi}v_2); \\ \beta \circ \bar{\psi}\colon \Pi \to \mathbf{U}(1), \quad l \mapsto \beta(\bar{\psi}(l)). \end{gathered} \tag{36}$$

In light of the non-degeneracy of $\mathbf{H}_A$, this implies that

$$\ker(\mathbf{H}_A \circ \psi) = \ker(\bar{\psi}) \subset \ker(H_A \circ \bar{\psi}) \subset V. \tag{37}$$

Now let (H_A, β) be the A.-H. data on (W, Γ) such that $\mathcal{M}$ is isomorphic to $\mathcal{L}(H_A, \beta)$. In light of (36), $\mathcal{F}$ is isomorphic to $\mathcal{L}(H_A \circ \bar{\psi}, \beta \circ \bar{\psi})$. In particular, $\mathcal{L} = \psi^*\mathcal{L}(\mathbf{H}_A, \mathbf{1}_\Gamma)$ is isomorphic to $\mathcal{L}(\mathbf{H}_A \circ \bar{\psi}, \mathbf{1}_\Pi)$. Here

$$\mathbf{1}_\Pi = \mathbf{1}_\Gamma \circ \bar{\psi}\colon \Pi \to \{1\} \subset \mathbf{U}(1)$$

is the trivial character of Π. Since $\mathcal{L}$ admits a non-zero holomorphic section, the inclusion (37) allows us to apply Corollary 10.3 to $\mathcal{N} = \mathcal{F}$ and $H_0 = H_A \circ \bar{\psi}$ and conclude that $\operatorname{Bim}(\overline{\mathcal{F}})$ is *not* Jordan.

Remark 10.7. Let V, Π, T, and $\mathcal{F}$ be as in Theorem 10.6. Suppose that $\mathcal{F} \cong \mathcal{L}(H, \alpha)$. Let $\alpha'\colon \Pi \to \mathbf{U}(1)$ be a map such that (H, α') also is A.-H. data on (V, Π). Let $\mathcal{F}'$ be a holomorphic line bundle on T that is isomorphic to $\mathcal{L}(H, \alpha')$. Then the same arguments as in the proof of Theorem 10.6 prove that $\operatorname{Bim}(\overline{\mathcal{F}'})$ is also non-Jordan (see Theorem 1.8 of [87]).

Chapter 4. Non-trivial $\mathbb{P}^1$-bundles over a non-uniruled base

In this chapter we consider the group $\operatorname{Aut}(X)$ for a non-trivial $\mathbb{P}^1$-bundle over a non-uniruled compact complex connected Kähler manifold Y. Recall that there is a homomorphism $\tau\colon \operatorname{Aut}(X) \to \operatorname{Aut}(Y)$ and its kernel is denoted by $\operatorname{Aut}(X)_p$. First we classify automorphisms $f \in \operatorname{Aut}(X)_p$, that is, those automorphisms that do not move fibres of p. We get that if $\operatorname{Aut}(X)_p \neq \{\mathrm{id}\}$, then either X or its double cover is a projectivization $\mathbb{P}(\mathcal{E})$ of rank 2 vector bundle over Y or its double cover, respectively. Thus, if Y is Kähler, then X is too [82; Proposition 3.5]. Thus the group $\operatorname{Aut}(X)$ is Jordan by a theorem of J. H. Kim [34]. It appears that if X is scarce (that is, does not have many sections: see Definition 11.5 below), then $\operatorname{Aut}_0(X)$ is commutative and $\operatorname{Aut}(X)$ is very Jordan. This is, for example, the case when Y is a torus of algebraic dimension zero.

11. Automorphisms of $\mathbb{P}^1$-bundles that preserve fibres

This section contains the classification of those automorphisms of a $\mathbb{P}^1$-bundle X that preserve the fibres of $p\colon X \to Y$. There are three different types; each type is described in a separate subsection.

Let (X, p, Y) be a $\mathbb{P}^1$-bundle over a compact complex connected manifold Y, that is,

- X, Y are compact connected complex manifolds of positive dimension;
- $p\colon X \to Y$ is a surjective holomorphic map;
- X is a holomorphically locally trivial fibre bundle over Y with fibre $\mathbb{P}^1$ and with the corresponding projection map $p\colon X \to Y$.

Let P_y stand for the fibre $p^{-1}(y)$. Let $U \subset Y$ be an open non-empty subset of Y. We call a covering $U = \bigcup U_i$ of Y by open subsets U_i, $i \in I$, to be *fine* if for every $i \in I$ there exists an isomorphism $\phi_i\colon V_i = p^{-1}(U_i) \to U_i \times \mathbb{P}^1_{(x_i:y_i)}$ such that

- (u, z_i), $u \in U_i$, $z_i = x_i/y_i \in \overline{\mathbb{C}}$, are local coordinates in $V_i := p^{-1}(U_i) \subset X$;
- $\mathbf{pr} \circ \phi_i = p$, where $\mathbf{pr}\colon U_i \times \mathbb{P}^1 \to \mathbb{P}^1$ is the natural projection (see (NA.14) in § 3).

Definition 11.1. A k-section S of p is a codimension 1 irreducible analytic subset $D \subset X$ such that the intersection $X \cap P_y$ is finite for every $y \in Y$ and consists of k distinct points for a general $y \in Y$. By a *bisection* we mean a 2-section that meets *every* fibre at two distinct points. Obviously, an *ordinary* holomorphic section S of p is a 1-section. A section S is defined by the set $\mathbf{a} = \{a_i(y)\}$ of functions $a_i\colon U_i \to \mathbb{P}^1$ such that $p(y, a_i(y)) = \mathrm{id}$, $y \in U_i$. We will denote this by $S = \mathbf{a}$.

Lemma 11.2. *Let A_1, A_2, A_3 be three distinct almost sections of p (see Definition 6.5). Assume that there is an analytic subspace $\Sigma \subset Y$ of codimension at least 2 such that the A_k, $k = 1, 2, 3$, are pairwise disjoint in $V = p^{-1}(U)$, where $U = Y \setminus \Sigma$.*

Then there exists an isomorphism $\Phi\colon X \to Y \times \mathbb{P}^1$ such that $\mathbf{pr} \circ \Phi = p$, where $\mathbf{pr}\colon Y \times \mathbb{P}^1 \to \mathbb{P}^1$ is the natural projection (see (NA.14) in § 3).

Proof. Indeed, let $\{U_i\}$ be a fine covering of Y and let

$$a_{ki}(u)x_i - b_{ki}(u)y_i = 0, \qquad u \in U_i,$$

be the equation of $A_k \cap U$, $k = 1, 2, 3$, over U_i. We define a meromorphic function $F(x)$ in every V_i by

$$F(x) = \frac{[a_{1i}(u)x_i - b_{1i}(u)y_i]\,[a_{2i}(u)b_{3i}(u) - a_{3i}(u)b_{2i}(u)]}{[a_{2i}(u)x_i - b_{2i}(u)y_i]\,[a_{1i}(u)b_{3i}(u) - a_{3i}(u)b_{1i}(u)]}, \qquad u = p(x). \tag{38}$$

Then $F(x)$ is globally everywhere defined and meromorphic in V. Its restrictions to $A_1 \cap V$, $A_2 \cap V$, and $A_3 \cap V$ are equal to 0, ∞, and 1, respectively.

The fibre of p has dimension 1, thus $X \setminus V = p^{-1}(\Sigma)$ has codimension 2 in X. Hence the function F can be extended to a meromorphic function on the whole of X by Levi's continuation theorem (Theorem 5.9). Thus, we have the bimeromorphic map $\Phi\colon X \to Y \times \mathbb{P}^1$, $\Phi(x) = (p(x), F(x))$, which induces an isomorphism of V onto $U \times \mathbb{P}^1$ that is compatible with p. According to Lemma 5.12, Φ is an isomorphism.

Remark 11.3. In particular, if there are three disjoint sections in X, then $X \sim Y \times \mathbb{P}^1$.

Remark 11.4. Note that a section is an almost section. If A is an almost section but not a section, then the set

$$\Sigma(A) = \{y \in Y \mid p^{-1}(y) \subset A\} \subset Y$$

has codimension at least 2 because

(a) $\widetilde{\Sigma} := p^{-1}(\Sigma(A))$ is a proper analytic subset of A with $\dim(A) = \dim(Y) = n$; thus $\dim(\widetilde{\Sigma}) \leqslant n-1$;

(b) every fibre of restriction of p to $\widetilde{\Sigma}$ has dimension 1.

Definition 11.5. We say that three sections S_1, S_2, and S_3 in X are a good configuration if $S_1 \cap S_2 = S_1 \cap S_3 = \varnothing$ and $S_2 \cap S_3 \neq \varnothing$. We say that three almost sections A_1, A_2, and A_3 in X are a special configuration if $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3$. We say that X is *scarce* if X admits no special configurations.

Lemma 11.6. *Let S_1, S_2, S_3, and S_4 be four distinct sections of p such that $S_1 \cap S_2 = \varnothing$ and $S_3 \cap S_4 = \varnothing$. Then $X \sim Y \times \mathbb{P}^1$.*

Proof. If $S_3 \cap (S_1 \cup S_2) = \varnothing$, then $X \sim Y \times \mathbb{P}^1$ (Remark 11.3). Assume that $X \not\sim Y \times \mathbb{P}^1$. Let $\varnothing \neq S_3 \cap S_2 = D \subset S_2$. Let $\{U_i\}_{i \in I}$ be a fine covering of Y. In every $V_i = p^{-1}(U_i)$ we choose coordinates (y, z_i) in such a way that $S_2 \cap V_i = \{z_i = 0\}$ and $S_1 \cap V_i = \{z_i = \infty\}$. Then $z_j = \lambda_{ij} z_i$ in $V_i \cap V_j$, where the λ_{ij} are holomorphic functions not vanishing in $U_i \cap U_j$.

Let $S_3 \cap V_i = \{(y, z_i = p_i(y)),\ y \in U_i\}$, where $p_j = \lambda_{ij} p_i$, and $S_4 \cap V_i = \{(y,\ z_i = q_i(y)),\ y \in U_i\}$, where $q_j = \lambda_{ij} q_i$. Then $r(y) := p_i(y)/q_i(y)$ is a globally defined meromorphic function on Y which omits value 1 (since $S_3 \cap S_4 = \varnothing$). Thus, $r := r(y) = \text{const}$. But then q_i vanishes at D and $S_3 \cap S_4 \supset D$. Contradiction.

Remark 11.7. We have also proved the following fact: If X contains two disjoint sections S_1 and S_2, then

- there is a holomorphic line bundle $\mathcal{L} := \mathcal{L}(S_1, S_2)$ such that $X \sim \mathbb{P}(\mathcal{L} \oplus \mathbf{1}_Y)$;
- there is a fine covering $\bigcup_{i \in I} U_i$ of Y and coordinates (u, z_i), $u \in U_i$, $z_i \in \overline{\mathbb{C}}$, in V_i, such that

$$S_1 \cap V_i = \{z_i = \infty\}, \qquad S_2 \cap V_i = \{z_i = 0\};$$

- $z_j = a_{ij} z_i$, and the cocycle $\mathbf{a} = \{a_{ij}\}$ defines $\mathcal{L}$.

Lemma 11.8. *If there exist three distinct almost sections A_1, A_2, and A_3 of p, then there exists a bimeromorphic map $\Phi\colon X \to Y \times \mathbb{P}^1$ such that $\mathbf{pr} \circ \Phi = p$.*

Proof. We keep the notation of the proof of Lemma 11.2.

Let

$$\Sigma(A_i) = \{y \in Y \mid p^{-1}(y) \subset A_i\}, \quad i = 1,2,3, \quad \text{and} \quad \Sigma = \bigcup_1^3 \Sigma(A_i).$$

Let $\widetilde{\Sigma} = p^{-1}(\Sigma)$.

The function $F(x)$ defined by (38) is defined and meromorphic at every point outside the set

$$D = (A_1 \cap A_3) \cup (A_2 \cap A_3) \cup (A_1 \cap A_2) \cup \widetilde{\Sigma}.$$

Since codimension of D is at least 2, the function F can be extended to a meromorphic function on X by Levi's theorem. Consider a map $\Phi\colon X \to Y \times \mathbb{P}^1$, $x \mapsto (p(x), F(x))$. It is meromorphic and induces an isomorphism on every fibre P_u, $u \notin p(D)$, to $\mathbb{P}^1$. Thus Φ is bimeromorphic.

LEMMA 11.9. *If X admits a good configuration S_1, S_2, S_3, then X admits a special configuration.*

PROOF. By assumption $S_1 \cap S_2 = S_1 \cap S_3 = \varnothing$ and $S_3 \cap S_2 \neq \varnothing$. Recall that S_2 is the zero section of the line bundle $\mathcal{L}(S_1, S_2)$ (see Remark 11.7). Let $\{U_i\}_{i\in I}$ be a fine covering of Y and (u, z_i), $u \in U_i$, $z_i \in \overline{\mathbb{C}}$, be coordinates in V_i such that $S_1 \cap V_i = \{z_i = \infty\}$ and $S_2 \cap V_i = \{z_i = 0\}$. Let the non-zero section of $\mathcal{L}$, namely, S_3, have the equation $z_i = h_i(u)$ in V_i. For any $c \in \mathbb{C}^*$ the equations $z_i = ch_i$ also define a section $S_4 \neq S_3$ of $\mathcal{L}$. By construction $S_2 \cap S_3 = S_2 \cap S_4 = S_3 \cap S_4 = \bigcup_{i\in I} \{h_i = 0\}$. Thus, S_2, S_3, and S_4 is a special configuration.

Now we consider the subgroup $\mathrm{Aut}(X)_p$ of those automorphisms f of X that do not move fibres of p, that is, such that $p \circ f = f$. Similarly to Lemma 11.2, every $f \in \mathrm{Aut}(X)_p$ defines locally a holomorphic map $\psi_f\colon Y \to \mathrm{PSL}(2,\mathbb{C})$ and the function

$$\mathrm{TD}(y), \quad y \mapsto \mathrm{TD}(\psi_f(y)) = \frac{\mathrm{tr}^2(\psi_f(y))}{\det(\psi_f(y))}$$

(see (NA.15) in § 3) is everywhere defined and holomorphic, hence is a constant on Y (see [7; Remark 4.9]). We denote this constant by $\mathrm{TD}(f)$.

Assume that $X \not\sim Y \times \mathbb{P}^1$. Let $f \in \mathrm{Aut}(X)_p$ and $f \neq \mathrm{id}$. Recall that $\mathrm{Fix}(f)$ is the set of all fixed points of f. Let $\{U_i\}_{i\in I}$ be a fine covering of Y. We summarize in Lemmas 11.10 and 11.11 below the properties of *non-identity* automorphisms $f \in \mathrm{Aut}(X)_p$ with $\mathrm{TD}(f) \neq 4$ (see [7]).

LEMMA 11.10. *Assume that (X, p, Y) is a $\mathbb{P}^1$-bundle and $X \not\sim Y \times \mathbb{P}^1$. Let $f \in \mathrm{Aut}(X)_p$, where $f \neq \mathrm{id}$ and $\mathrm{TD}(f) \neq 4$. Then one of following two cases holds.*

Case A: *the set* $\mathrm{Fix}(f)$ *consists of exactly two disjoint sections S_1, S_2 of p. We say that f is of type* **A** *with data (S_1, S_2), an ordered pair. In the notation of Remark* 11.7, *let $\{U_i\}_{i\in I}$, $\mathcal{L}(S_1, S_2)$, and $\mathbf{a} = \{a_{ij}\}$ be the corresponding fine covering, holomorphic line bundle, and cocycle, respectively. Then*

1) *there is a number $\lambda_f \in \mathbb{C}^*$ such that in every V_i*

$$f(u, z_i) = (u, \lambda_f z_i); \tag{39}$$

2) *if $G_0 \subset \mathrm{Aut}(X)_p$ is the subgroup of all $f \in \mathrm{Aut}(X)_p$ such that $f(S_1) = S_1$, $f(S_2) = S_2$, then $G_0 \cong \mathbb{C}^*$;*

3) *the restriction $f \to f|_{P_y}$ defines a group embedding of G_0 into* $\mathrm{Aut}(P_y)$.

Case C: *the set* $\mathrm{Fix}(f)$ *is a smooth unramified double cover S of Y.* (*We call such an automorphism f an automorphism of type* **C** *with data S.*) *Here S is a bisection of p.*

PROOF. $\mathrm{TD}(f) \neq 4$ implies that f has exactly two distinct fixed points in *every* fibre $P_y = p^{-1}(y)$, $y \in Y$. Thus $\mathrm{Fix}(f)$ is either a union of two disjoint sections or a 2-section of p. In case A equality (39) follows from the fact that

$$f(u, z_i) = \lambda_i z_i, \qquad f(u, z_j) = \lambda_j z_j = \lambda_j a_{ij} z_i = a_{ij} \lambda_i z_i.$$

The constant $\lambda_f = \lambda_i \neq 0$ does not depend on the choice of the fibre, hence f is determined uniquely by its restriction to any given fibre. On the other hand, for every $\lambda \in \mathbb{C}^*$ there exists an automorphism $f_\lambda \in \mathrm{Aut}(X)_p$ defined in every V_i by

$$(u, z_i) \to (u, \lambda z_i).$$

Therefore, $G_0 \cong \mathbb{C}^*$. The lemma is proved.

LEMMA 11.11 (see [7]). *Let S be a bisection of the $\mathbb{P}^1$-bundle (X, p, Y). Consider*

$$\widetilde{X} := \widetilde{X}_S := S \times_Y X = \{(s, x) \in S \times X \subset X \times X \mid p(s) = p(x)\}.$$

We denote the restriction of p to S by the same letter p, and let p_X and $\tilde{p}$ stand for the restrictions to $\widetilde{X}$ of the natural projections $S \times X \to X$ and $S \times X \to S$, respectively. We write $\mathrm{inv}\colon S \to S$ *for the involution (the only non-trivial deck transformation for $p|_S$). Then $(\widetilde{X}, \tilde{p}, S)$ is a $\mathbb{P}^1$-bundle with the following properties.*

(a) *The diagram*

$$\begin{array}{ccc} X & \xrightarrow{p_X} & X \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ S & \xrightarrow{p|_S} & Y \end{array} \tag{40}$$

commutes.

(b) *$p_X\colon \widetilde{X} \to X$ is an unramified double cover of X.*

(c) *Every fibre $\tilde{p}^{-1}(s)$, $s \in S$, is isomorphic to*

$$P_{p(s)} = p^{-1}(p(s)) \sim \mathbb{P}^1.$$

(d) *The $\mathbb{P}^1$-bundle $\widetilde{X}$ over S has two disjoint sections, namely,*

$$S_+ := S_+(f) := \{(s, s) \in \widetilde{X} \mid s \in S \subset X\},$$
$$S_- := S_-(f) := \{(s, \mathrm{inv}(s)) \in \widetilde{X} \mid s \in S \subset X\}.$$

They are mapped onto S isomorphically by p_X.

(e) *Every $h \in \mathrm{Aut}(X)_p$ induces an automorphism $\tilde{h} \in \mathrm{Aut}(\widetilde{X})_{\tilde{p}}$ defined by*

$$\tilde{h}(s, x) = (s, h(x)).$$

(f) *The involution $s \mapsto \mathrm{inv}(s)$ can be extended from S to a holomorphic involution of $\widetilde{X}$ by*

$$\mathrm{inv}(s, x) = (\mathrm{inv}(s), x).$$

(g) *Every section $N = \{y, \sigma(y)\}$ of p in X induces the section $\widetilde{N} := \{(s, \sigma(p(s)))\}$ of $\tilde{p}$ in $\widetilde{X}$. We have $p_X(\widetilde{N}) = N$ is a section of p, thus $\widetilde{N}$ cannot coincide with S_+ or S_-.*

11.1. Automorphisms with $\mathrm{TD} = 4$**.** If $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$, and $\mathrm{TD}(f) = 4$, then there is precisely one fixed point of f in the fibre $P_y = p^{-1}(y)$ over the general point $y \in Y$. This means that $\mathrm{Fix}(f)$ contains precisely one almost section D of p. In this case we say that f is of type **B** with data D.

LEMMA 11.12. *Let (X, p, Y) be a $\mathbb{P}^1$-bundle, where X and Y are compact connected complex manifolds*, $\dim(Y) = n$, $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$, *and* $\mathrm{TD}(f) = 4$. *Let D be the only almost section contained in* $\mathrm{Fix}(f)$. *Let $\Sigma = \{y \in Y \mid P_y \subset D\}$ and $U = Y \setminus \Sigma$, $V = p^{-1}(U) \subset X$. Let $\widetilde{S}$ be the union of all irreducible components of* $\mathrm{Fix}(f)$ *distinct from D, and let $S = p(\widetilde{S})$.*

Then

(i) *there is a fine covering $\{U_i\}_{i \in J}$ of U and coordinates (u, z_i) in $V_i = p^{-1}(U_i)$ such that $D \cap V_i = \{z_i = \infty\}$;*

(ii) *$f(u, z_i) = (u, z_i + \tau_i(u))$, where the τ_i are holomorphic functions on U_i;*

(iii) *if $i, j \in J$, then $z_j = \mu_{ij} z_i + \nu_{ij}$ where μ_{ij} and ν_{ij} are holomorphic functions in $U_i \cap U_j$ and μ_{ij} does not vanish. Moreover, the μ_{ij} depend on D and the choice of coordinates in V_i but not on f;*

(iv) *if $i, j \in J$, then $\tau_j = \mu_{ij} \tau_i$ in $U_i \cap U_j$;*

(v) *S has pure codimension 1 in Y.*

PROOF. Recall that the set Σ has codimension at least 2 in Y (Remark 11.4). Now we prove the assertions of the lemma.

(i) follows from the fact that D is a section of p over U.

(ii) follows from the fact that $D \subset \mathrm{Fix}(f)$, thus the restriction of f onto a fibre P_y, $y \in U_i$, is an automorphism of $\mathbb{P}^1$ which is either identity or has the only fixed point $z_i = \infty$.

(iii) follows from the fact that z_j is obtained from z_i by an automorphism of $\mathbb{P}^1$ with $z = \infty$ fixed.

Since X admits an almost section, we see that $X \sim \mathbb{P}(\mathcal{E})$ for some rank 2 holomorphic vector bundle $\mathcal{E}$ on Y with projection $\pi \colon \mathcal{E} \to Y$ ([78; Lemma 3.5] and Theorem 6.7). This means that we have a fine covering $\{U_i\}$ and a cocycle $A_{ij} \in \mathrm{GL}(2, \mathcal{O}(U_i \cap U_j))$ of two by two transition matrices for $\mathcal{E}$ such that

- $\pi^{-1}(U_i) \sim U_i \times \mathbb{C}^2_{x_i, y_i}$;
- if $U_i \cap U_j \neq \varnothing$, then
$$A_{ij} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} x_j \\ y_j \end{bmatrix}.$$

Since $D \cap V$ is a section of p over U, we can choose a basis in $\mathbb{C}^2_{x_i, y_i}$ in such a way that the preimage of $D \cap U_i$ in $U_i \times \mathbb{C}^2_{x_i, y_i}$ is $U_i \times \{(x_i, 0)\}$, $x_i \in \mathbb{C}$.

- For these coordinates we have
$$A_{ij} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_{i,j} \\ 0 \end{bmatrix}.$$
- Moreover,
$$A_{ij} = \begin{bmatrix} \lambda_{i,j} & b_{ij} \\ 0 & \tilde{\lambda}_{i,j} \end{bmatrix}, \tag{41}$$
where b_{ij}, $\lambda_{i,j}$, $\tilde{\lambda}_{i,j}$, and the functions
$$d_{ij} = \lambda_{i,j} \tilde{\lambda}_{i,j} = \det(A_{ij}) \tag{42}$$
are holomorphic in $U_i \cap U_j$.

Now let $z_j = x_j/y_j$, $z_i = x_i/y_i$. Then

$$z_j = \frac{\lambda_{i,j}x_i + b_{ij}y_i}{y_i\tilde{\lambda}_{i,j}} = \mu_{ij}z_i + \nu_{ij}. \tag{43}$$

Thus $\mu_{ij} = \lambda_{i,j}^2/d_{ij} = \lambda_{i,j}/\tilde{\lambda}_{i,j}$ depends on the choice of D and is defined by the eigenvalue of the basis vector in the invariant subspace representing D. It does not depend on the choice of f with the given data D.

Note that both $\{\lambda_{i,j}\}$ and $\{\tilde{\lambda}_{i,j}\}$ form cocycles for a covering of U.

Item (iv) follows from the fact that f is globally defined and D is fixed, thus

$$f(u, z_j) = (u, z_j + \tau_j(u)) = (u, \mu_{ij}z_i + \nu_{ij} + \tau_j(u)) = (u, \mu_{ij}(z_i + \tau_i(u)) + \nu_{ij}).$$

Item (v) follows from the fact that the functions τ_i are holomorphic and $S \cap U_i = \{\tau_i = 0\}$. Indeed, let $\widetilde{S}_1 \subset \widetilde{S}$ be an irreducible component of $\widetilde{S}$. It cannot be an almost section, thus $S_1 = p(\widetilde{S}_1)$ is a proper analytic subset of Y. Moreover, since $\widetilde{\Sigma} \subset D$, we have: $\widetilde{S}_1 \not\subset \widetilde{\Sigma}$, $S_1 \not\subset \Sigma$. Thus, $S_1 \cap U$ is a dense open subset of S_1. Since $S \cap U_i = \{\tau_i = 0\}$ has pure codimension 1 (if $S \cap U_i \neq \varnothing$), the same is valid for every component of it that intersects U_i. Thus, $\dim(S_1) = n - 1$.

Lemma 11.12 is proved.

Proposition 11.13. *In the notation of Lemma 11.12 let $S_1, \dots, S_k$ be all the irreducible components of S. Then*

(i) *For every l, $1 \leqslant l \leqslant k$, a non-negative number n_l is defined that is the order of the zero of τ_i along the component S_l if $S_l \cap U_i \neq \varnothing$. It depends on l but not on i. The holomorphic line bundle $\mathcal{L}(f)$ corresponding to the effective divisor $\Delta_f := \sum_{l=1}^{k} n_l S_l$ restricts to U to the holomorphic line bundle defined by the cocycle μ_{ij}.*

(ii) *Let G_D be the subgroup of $\operatorname{Aut}(X)_p$ of all those $g \in \operatorname{Aut}(X)_p$, that have $\operatorname{TD}(g) = 4$ and $D \subset \operatorname{Fix}(g)$. Then G_D is isomorphic to the additive group of $H^0(Y, \mathcal{L}(f))$. Thus $G_D \cong (\mathbb{C}^+)^n$, $n > 0$.*

Proof. Let S_l be an irreducible component of S. For every U_i such that $S_l \cap U_i \neq \varnothing$ the order n_{li} of zero of τ_i along S_l is defined. In $U_i \cap U_j$ we have $\tau_j = \tau_i\mu_{ij}$. Since μ_{ij} does not vanish, τ_j has the same order of zero along $S_l \cap U_j$. Since S_l is irreducible and $U \cap S_l$ is open and dense in S_l, the order n_l is well defined (see, for example, [31; Remarks 2.3.6]). By construction, the divisor of τ_i in U_i is $\Delta_f \cap U_i$, thus the transition function for $\mathcal{L}(f)$ in $U_i \cap U_j$ is $\tau_j/\tau_i = \mu_{ij}$.

Let $h \in \operatorname{Aut}(X)_p$, and $\operatorname{TD}(h) = 4$, and $D \subset \operatorname{Fix}(g)$. Applying Lemma 11.12(iii) we get that $h(u, z_i) = (u, z_i + h_i(u))$, where $h_j = \mu_{ij}h_i$. Thus the function defined in every U_i by $G_h(u) = \dfrac{h_i}{\tau_i}$ is meromorphic in U. By Levi's theorem $G_h(u)$ is meromorphic on Y. By construction its divisor $(G_h) \geqslant -\Delta_f$, thus $G \in H^0(Y, \mathcal{L}(f))$.

On the other hand let G be a meromorphic function on Y with divisor $(G) \geqslant -\Delta_f$ (that is, $G \in H^0(Y, \mathcal{L}(f))$). For every i the function $h_i = G\tau_i$ is holomorphic in U_i, hence we can define a holomorphic automorphism of every $V_i = p^{-1}(U_i)$ by

$$h(u, z_i) = (u, z_i + h_i(u)). \tag{44}$$

Since $h_j := \mu_{ij} h_i$, the map h is an automorphism of V. Moreover, all the points of $D \cap V = \bigcup\{z_i = \infty\}$ are fixed by h. By Lemma 5.13 it can be extended to a bimeromorphic map of X.

By Lemma 5.12, $h \in \mathrm{Aut}(X)_p$. Moreover, $\mathrm{Fix}(\tilde{h})$ contains the closure of $D \cap V$, that is, D. In the general fibre P_y of p it has precisely one fixed point $D \cap P_y$, thus $\mathrm{TD}(h) = 4$.

Hence we obtain a one-to-one map

$$\phi\colon G_D \to H^0(Y, \mathcal{L}(f)), \quad h \in G_D \mapsto G_h \in H^0(Y, \mathcal{L}(f)).$$

From Lemma 11.12(iii) we get that the composition of $g, h \in \mathrm{Aut}(X)_p$ is defined by the cocycle $g_i + h_i$ made up of the corresponding cocycles, which implies that

$$\phi(h \circ g) = \phi(h) + \phi(g).$$

This proves the proposition.

The next lemma answers the question when an almost section $D \subset \mathrm{Fix}(f)$ is a section. We used this fact in [7], in dealing with automorphisms of type **B**.

LEMMA 11.14. *In the notation of Lemma 11.12 and Proposition 11.13, if $\Delta_f = 0$, then D is a section.*

PROOF. First note that $\Delta_f = 0$ implies that the corresponding line bundle $\mathcal{L}_f$ is trivial and $f \neq \mathrm{id}$ in the fibre $F_y = p^{-1}(y)$ if $y \notin \Sigma$.

Since X admits an almost section, $X \sim \mathbb{P}(\mathcal{E})$ for some rank 2 holomorphic vector bundle $\mathcal{E}$ on Y ([78; Lemma 3.5], Theorem 6.7). This means that we have a fine covering $\{U_i\}_{i \in I}$ of Y and a cocycle A_{ij} of 2×2 matrices (with entries holomorphic in $U_i \cap U_j$) such that

(a) $p^{-1}(U_i) = V_i \sim U_i \times \mathbb{P}^1_{x_i:y_i}$, $z_i = x_i/y_i$ and if $U_i \cap U_j \neq \varnothing$, then

$$A_{ij} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} x_j \\ y_j \end{bmatrix};$$

(b) in every U_i there exists a 2×2 matrix F_i (representing f) whose entries are holomorphic functions (in $u \in U_i$) and such that $\mathrm{TD}(F_i) = 4$, $\det(F_i) = d_i \neq 0$, and $f(u, (x_i : y_i)) = (u, (x_i' : y_i'))$, where

$$\begin{bmatrix} x_i' \\ y_i' \end{bmatrix} = F_i \begin{bmatrix} x_i \\ y_i \end{bmatrix};$$

(c) $F_j(u) A_{ij}(u) = A_{ij}(u) F_i(u) \dfrac{d_j}{d_i}$.

Since $4d_i = \mathrm{tr}(F_i)^2$ is a square, we can divide F_i by $\mathrm{tr}(F_i)/2 = \sqrt{d_i}$ and assume that $d_i = 1$ (we use that $(x_i : y_i)$ are homogeneous coordinates in $\mathbb{P}^1_{x_i:y_i}$).

Assume that D is not a section, that is, $\Sigma = \{y \in Y \mid p^{-1}(y) \subset D\} \neq \varnothing$.

Let the fine covering of Y consist of open sets $U_0, \dots, U_N$, and let $U_0, \dots, U_k$ meet Σ, while $U = Y \setminus \Sigma = \bigcup_{i=k+1}^{N} U_i$.

Then for each $i > k$ we may assume the following.

1) $F_i = \begin{bmatrix} 1 & \tau_i \\ 0 & 1 \end{bmatrix} = I + \tau_i V$ where I is the identity matrix, the τ_i are holomorphic functions in U_i, and $V = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (by Lemma 11.12(ii)).
2) Recall that $\mathcal{L}_f\big|_U$ is defined on U by a cocycle $\{\mu_{ij}\}$, where $\mu_{ij} = \tau_j/\tau_i$ is a non-vanishing holomorphic function on $U_i \cap U_j$ if $U_i \subset U$ and $U_j \subset U$ (by Lemma 11.12). Since $\mathcal{L}_f$ is trivial, we may assume that the cocycle $\{\mu_{ij}\}$ is trivial, that is, $\mu_{ij} = 1$ and $\tau_i = 1$ do not depend on i for $U_i \subset U = Y \setminus \Sigma$. Moreover, from (42) and (43) we get that the A_{ij} are triangular matrices, and for the eigenvalues λ_{ij}, $\tilde{\lambda}_{ij}$ of the A_{ij} we have $\lambda_{ij} = \tilde{\lambda}_{ij}$. Hence $\det(A_{ij}) = \lambda_{ij}^2$.
   Thus, if both $i, j > k$, we may assume that

$$A_{ij} = \begin{bmatrix} \lambda_{ij} & \nu_{ij} \\ 0 & \lambda_{ij} \end{bmatrix}$$

   where λ_{ij}, ν_{ij} are holomorphic functions in $U_i \cap U_j$.

Take a point $\mathbf{s} \in \Sigma$ and let U_0 be a neighbourhood of $\mathbf{s}$. Let $\tilde{r}(\mathbf{s})$ be the number of those neighbourhoods U_i with $i > k$ in our fine covering for which $U_i \cap U_0 \neq \varnothing$. Let $r = \tilde{r}(\mathbf{s})$. Let

$$U_t, \dots, U_{t+r}, \qquad t > k,$$

be those neighbourhoods for which $U_i \cap U_0 \neq \varnothing$, $t \leqslant i \leqslant t + r$. For $t \leqslant i, j \leqslant t + r$ we have the following:

(a) $F_0 = A_{i0}(u) F_i A_{i0}(u)^{-1} = I + W_i = I + A_{j0}(u) V A_{j0}(u)^{-1} = I + W_j$, where $W_i = A_{i0}(u) V A_{i0}(u)^{-1}$, $t \leqslant i \leqslant t+r$; it follows that the matrix function W_i defined *a priori* in $U_0 \cap U_i$ can be extended to a matrix function with holomorphic entries to $U_0 \setminus \Sigma$, hence, to the whole of U_0 and

$$W_i = W_j; \tag{45}$$

(b) if $U_i \cap U_j \cap U_0 \neq \varnothing$, then

$$A_{i0}(u) A_{j0}^{-1}(u) = A_{ij}(u); \tag{46}$$

(c) if we let

$$A_{i0}(u) = \begin{bmatrix} \alpha_1(u) & \beta_1(u) \\ \gamma_1(u) & \delta_1(u) \end{bmatrix} \quad \text{and} \quad A_{j0}(u) = \begin{bmatrix} \alpha_2(u) & \beta_2(u) \\ \gamma_2(u) & \delta_2(u) \end{bmatrix},$$

then

$$W_i(u) = \begin{bmatrix} -\alpha_1(u)\gamma_1(u) & \alpha_1^2(u) \\ -\gamma_1^2(u) & \alpha_1(u)\gamma_1(u) \end{bmatrix} = W_j(u) = \begin{bmatrix} -\alpha_2(u)\gamma_2(u) & \alpha_2^2(u) \\ -\gamma_2^2(u) & \alpha_2(u)\gamma_2(u) \end{bmatrix} \tag{47}$$

and

$$A_{i0}(u) A_{j0}^{-1}(u) = \frac{1}{d_{j0}} \begin{bmatrix} \alpha_1\delta_2 - \beta_1\gamma_2 & -\alpha_1\beta_2 + \beta_1\alpha_2 \\ \gamma_1\delta_2 - \delta_1\gamma_2 & -\gamma_1\beta_2 + \delta_1\alpha_2 \end{bmatrix} = \begin{bmatrix} \lambda_{ij} & \nu_{ij} \\ 0 & \lambda_{ij} \end{bmatrix}. \tag{48}$$

Let $\widetilde{U}_{ij} = U_i \cap U_j \cap U_0 \neq \varnothing$. From (47) we get that $\alpha_1^2 = \alpha_2^2$ and $\alpha_1(u)\gamma_1(u) = \alpha_2(u)\gamma_2(u)$ in $\widetilde{U}_{ij}$. Note that these equations are valid in the whole of U_0 since W_i and W_j are defined there.

In $\widetilde{U}_{ij}$ the following three cases are possible: $\alpha_1 = \alpha_2$ and $\gamma_1 = \gamma_2$, $\alpha_1 = -\alpha_2$ and $\gamma_1 = -\gamma_2$, or $\alpha_1 = \alpha_2 = 0$.

Case 1: $\alpha_1 = \alpha_2$ and $\gamma_1 = \gamma_2$ in $\widetilde{U}_{ij}$. Plugging this into (48) we obtain the following:

$$\frac{1}{d_{j0}}\begin{bmatrix}\alpha_1\delta_2 - \beta_1\gamma_2 & -\alpha_1\beta_2 + \beta_1\alpha_2\\ \gamma_1\delta_2 - \delta_1\gamma_2 & -\gamma_1\beta_2 + \delta_1\alpha_2\end{bmatrix} = \frac{1}{d_{j0}}\begin{bmatrix}\alpha_1\delta_2 - \beta_1\gamma_1 & -\alpha_1\beta_2 + \beta_1\alpha_1\\ \gamma_1\delta_2 - \delta_1\gamma_1 & -\gamma_2\beta_2 + \delta_1\alpha_2\end{bmatrix}$$
$$= \frac{1}{d_{j0}}\begin{bmatrix}d_{i0} + \alpha_1(\delta_2 - \delta_1) & \alpha_1(\beta_1 - \beta_2)\\ \gamma_1(\delta_2 - \delta_1) & d_{j0} - \alpha_2(\delta_2 - \delta_1)\end{bmatrix} = \begin{bmatrix}\lambda_{ij} & \nu_{ij}\\ 0 & \lambda_{ij}\end{bmatrix}.$$

Thus there are two cases again.

Case 1.1: $\gamma_1 \equiv 0$ in $\widetilde{U}_{ij}$, hence $\gamma_1^2 = 0$ in U_0. Then in the whole of U_0

$$F_0 = \begin{bmatrix}1 & \alpha_1^2(u)\\ 0 & 1\end{bmatrix}$$

and $\alpha_1^2(u)$ does not vanish in U_0 since $\operatorname{codim}(\Sigma) \leqslant 2$ and $\Delta_f = 0$, that is, $F_0(u) \neq \mathbf{I}$ if $u \notin \Sigma$. Thus, $D \cap V_0 = \{y_0 = 0\}$ and $\Sigma \cap U_0 = \varnothing$. This contradicts the inclusion $\mathbf{s} \in \Sigma$.

Case 1.2: $\gamma_1 \not\equiv 0$ and $\delta_2 \equiv \delta_1$ in $\widetilde{U}_{ij}$. Then $1 = \lambda_{ij} = d_{i0}/d_{j0}$. Moreover,

$$\beta_1 = \frac{\alpha_1\delta_1 - d_{i0}}{\gamma_1} = \beta_2 = \frac{\alpha_2\delta_2 - d_{j0}}{\gamma_2}$$

and $\nu_{ij} = 0$ in $\widetilde{U}_{ij} \cap \{\gamma_1 \neq 0\}$. Since this set is open in $U_i \cap U_j$, we have $\nu_{ij} \equiv 0$ and

$$A_{ij} \equiv \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

It follows that there is a compatible with p isomorphism $V_i \cup V_j \sim (U_i \cup U_j) \times \mathbb{P}^1_z$, where $z = x_i/y_i = x_j/y_j$. Thus we can replace U_i, U_j by $U_i \cup U_j$ and obtain a new fine covering of Y consisting of $N-1$ open subsets and such that $\tilde{r}(\mathbf{s}) = r-1$. Since U_0 is connected we can repeat this process (recall that $\gamma_1 = \gamma_2 \not\equiv 0$ in $U_i \cup U_j$ so we remain in Case 1.2) till we obtain a covering with $\tilde{r}(\mathbf{s}) = 1$.

Thus, since $U_0 \setminus \Sigma$ is contained in $U_t \cup \dots \cup U_{t+r}$ we get that $p^{-1}(U_0 \setminus \Sigma) \sim (U_0 \setminus \Sigma) \times \mathbb{P}^1_z$. By Lemma 5.12 and Lemma 5.13 this extends to an isomorphism and D is the preimage of $\{z = \infty\}$.

Case 2: $\alpha_1 = -\alpha_2$ and $\gamma_1 = -\gamma_2$. Plugging this into (48) we obtain the following:

$$\frac{1}{d_{j0}}\begin{bmatrix}\alpha_1\delta_2 - \beta_1\gamma_2 & -\alpha_1\beta_2 + \beta_1\alpha_2\\ \gamma_1\delta_2 - \delta_1\gamma_2 & -\gamma_1\beta_2 + \delta_1\alpha_2\end{bmatrix} = \frac{1}{d_{j0}}\begin{bmatrix}\alpha_1\delta_2 + \beta_1\gamma_1 & -\alpha_1\beta_2 - \beta_1\alpha_1\\ \gamma_1\delta_2 + \delta_1\gamma_1 & \gamma_2\beta_2 + \delta_1\alpha_2\end{bmatrix}$$
$$= \frac{1}{d_{j0}}\begin{bmatrix}-d_{i0} + \alpha_1(\delta_2 + \delta_1) & -\alpha_1(\beta_1 + \beta_2)\\ \gamma_1(\delta_2 + \delta_1) & -d_{j0} - \alpha_1(\delta_2 + \delta_1)\end{bmatrix} = \begin{bmatrix}\lambda_{ij} & \nu_{ij}\\ 0 & \lambda_{ij}\end{bmatrix}.$$

Similarly to Case 1 we have the two following options.

Case 2.1: $\gamma_1 \equiv 0$. Then

$$F_0 = \begin{bmatrix} 1 & \alpha_1^2(u) \\ 0 & 1 \end{bmatrix}$$

and D is a section of p over U_0.

Case 2.2: $\gamma_1 \not\equiv 0$ and $\delta_2 \equiv -\delta_1$ in $\widetilde{U}_{ij}$. Then $-1 = \lambda_{ij} = -d_{i0}/d_{j0}$.

Then $\beta_1 = (\alpha_1\delta_1 - d_{i0})/\gamma_1 = -\beta_2 = -(\alpha_2\delta_2 - d_{j0})/\gamma_2$ and $\nu_{ij} = 0$. Similarly to Case 1.2 we get that $p^{-1}(U_0 \setminus \Sigma) \sim (U_0 \setminus \Sigma) \times \mathbb{P}^1_z$ and D is a section of p over U_0.

Case 3: $\alpha_1 = \alpha_2 = 0$. According to (47)

$$F_0 = I + W_i = \begin{bmatrix} 1 & 0 \\ -\gamma_1^2(u) & 1 \end{bmatrix}$$

and $\gamma_1^2(u)$ does not vanish in U_0 since $\Delta_f = 0$. Thus $D \cap V_0 = \{z = 0\}$, which contradicts the inclusion $\mathbf{s} \in \Sigma$.

Lemma 11.14 is proved.

Remark 11.15. We may assume that a fine covering of Y contains a finite covering of U since $U_0 \setminus \Sigma$ can be covered by two neighbourhoods $U_0 \cap \{\alpha_i \neq 0\}$ and $U_0 \cap \{\gamma_i \neq 0\}$ (see (47)).

Lemma 11.16. *Let $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$, be an automorphism of type* **B** *with data D. Assume that there exists an almost section A of p distinct from D. Then X contains a special configuration.*

Proof. Since $A \neq D$, and $A \not\subset \mathrm{Fix}(f)$, we have $A_1 := f(A) \neq D$ and $A_1 \neq A$. Similarly, $A_2 := f(A_1) \neq D$ and $A_2 \neq A_1$. Let us show that $A_2 \neq A$.

If $A_2 = A$, then in the fibre $P_y = p^{-1}(y)$ over the general point $y \in Y$ there is point $a = A \cap P_y$ such that $f(a) \neq a$ but $f(f(a)) = a$. But along the general fibre P_y the map f acts as a translation $z \to z + \tau$ where $\tau \neq 0$. This map has no periodic points except $z \neq \infty$. This contradiction shows that $A_2 \neq A$.

Let us show that A, A_1, A_2 is a special configuration. For a fibre P_y we have the following options:

- $f|_{P_y} = \mathrm{id}$; then $P_y \cap A = P_y \cap A_1 = P_y \cap A_2$;
- $f|_{P_y}$ is translation $z \to z + \tau$ and $P_y \cap A \neq P_y \cap D$; then $P_y \cap A$, $P_y \cap A_1$, and $P_y \cap A_2$ are pairwise disjoint sets;
- $f|_{P_y}$ is translation $z \to z + \tau$ and $a := P_y \cap A = P_y \cap D$, then $P_y \cap A_1 = a$, $P_y \cap A_2 = a$.

It follows that $A \cap A_1 = A \cap A_2 = A_1 \cap A_2$ and A, A_1, A_2 is a special configuration.

Corollary 11.17. *In the notation of Lemma 11.16, if X is scarce and $\mathrm{Aut}(X)_p$ contains an automorphism f of type* **B** *with data D, then it contains no automorphisms of type* **B** *with data different from D, nor automorphisms of type* **A**.

Proof. Indeed, the existence of such automorphisms would imply the existence of an almost section (in particular, a section in the case of type **A**) distinct from the one contained in $\mathrm{Fix}(f)$.

11.2. Automorphisms of type A.

LEMMA 11.18. *Assume that $X \not\sim Y \times \mathbb{P}^1$. Let S_1 and S_2 be two sections of p such that $S_1 \cap S_2 = \varnothing$. Let $f \in \mathrm{Aut}(X)_p$. Then one of the following holds*:

(a) $f(S_1) \subset S_1 \cup S_2$;
(b) $f(S_2) \subset S_1 \cup S_2$;
(c) $f(S_1 \cup S_2) = S_1 \cup S_2$.

PROOF. Note that a fibrewise automorphism moves a section to a section. Let $S_3 = f(S_1)$, $S_4 = f(S_2)$. Since $S_1 \cap S_2 = \varnothing$, we have $S_3 \cap S_4 = \varnothing$. According to Lemma 11.6 this can occur only if the pairs (S_3, S_4) and (S_1, S_2) share a section. This may occur only if one of the sections of the pair (S_3, S_4) coincides with either S_1 or S_2. The lemma is proved.

Recall that the group G_0 of all those $f \in \mathrm{Aut}(X)_p$ that have the data (S_1, S_2) is isomorphic to $\mathbb{C}^*$ (see Lemma 11.10).

Assume that the holomorphic line bundle $\mathcal{L}(S_1, S_2)$ is defined by a cocycle $\{\lambda_{ij}\}$ and $\mathcal{L}(S_1, S_2)^{\otimes 2}$ has a section $T \subset X$ defined by $\mathbf{a} := \{a_i(y)\}$ such that $a_j(y) = \lambda_{ij}^2 a_i(y)$.

Define
$$\phi_T\colon X \to X, \quad \phi_T(y, z_i) = \left(y, \frac{a_i(y)}{z_i}\right).$$
The fixed point set $\mathrm{Fix}(\phi_T) = \{\phi_T(y, z_i) = (y, z_i)\}$ is defined by $T \cap V_i = \{z_i^2 = a_i\}$. If $\phi_T \in \mathrm{Aut}(X)_p$, then the a_i do not vanish. In this case $\mathbf{a} := \{a_i\}$ provide a section of $\mathcal{L}_p^{\otimes 2}$ that does not meet the zero section, thus $\mathcal{L}_p^{\otimes 2}$ is a trivial bundle and we can define z_i in such a way that $a_i = a = \mathrm{const} \neq 0$. Then we write $T = T_a$ and $\phi_a := \phi_T$.

PROPOSITION 11.19. *Let (X, p, Y) be a $\mathbb{P}^1$-bundle, where X and Y are compact connected complex manifolds, and $X \not\sim Y \times \mathbb{P}^1$. Let S_1 and S_2 be two sections of p such that $S_1 \cap S_2 = \varnothing$. Let $\mathcal{L} := \mathcal{L}(S_1, S_2)$ be the corresponding holomorphic line bundle over Y. Let*

- *$G_1 \subset \mathrm{Aut}(X)_p$ be the subgroup of all $f \in \mathrm{Aut}(X)_p$ such that $f(S_1) = S_1$;*
- *$G_2 \subset \mathrm{Aut}(X)_p$ be the subgroup of all $f \in \mathrm{Aut}(X)_p$ such that $f(S_2) = S_2$;*
- *$G \subset \mathrm{Aut}(X)_p$ be the subgroup of all $f \in \mathrm{Aut}(X)_p$ such that $f(S_1 \cup S_2) = S_1 \cup S_2$;*
- *F_1 be the additive group of $H^0(Y, \mathcal{O}(\mathcal{L}))$;*
- *F_2 be the additive group of $H^0(Y, \mathcal{O}(\mathcal{L}^{-1}))$.*

Then

1) *X does not admit a good configuration (see Definition 11.5) if and only if $F_1 = F_2 = \{0\}$;*
2) *$G_1 \cong \mathbb{C}^* \rtimes F_1$;*
3) *$G_2 \cong \mathbb{C}^* \rtimes F_2$;*
4) *either $G = G_0 = G_1 \cap G_2 \cong \mathbb{C}^*$ or $\mathcal{L}_p^{\otimes 2}$ is a trivial bundle and $G = G_0 \sqcup \phi_a \cdot G_0$ for some $a \in \mathbb{C}^*$.*

PROOF. Let $\lambda = \{\lambda_{ij}\}$ be the cocycle corresponding to $\mathcal{L}$. Take $f \in G_1$. Since $S_1 = \{z_i = \infty\}$ is f-invariant, we have
$$f(y, z) = (y, a_i z_i + b_i) \tag{49}$$

in V_i, where both a_i and b_i are holomorphic functions in U_i. Since f is globally defined, we have

$$\lambda_{ij}(a_i z_i + b_i) = a_j \lambda_{ij} z_i + b_j.$$

It follows that $a_i = a_j := a$ is constant (as globally defined holomorphic function) and $b_j = \lambda_{ij} b_i$, hence $\mathbf{b} := \{b_i\}$ is a section of $\mathcal{L}$. On the other hand every section $\mathbf{b} := \{b_i\}$ of $\mathcal{L}$ defines $f \in G_1$ by formula (49). Thus, G_1 is isomorphic to the group of matrices

$$\begin{bmatrix} a & \mathbf{b} \\ 0 & 1 \end{bmatrix},$$

where $a \in \mathbb{C}^*$ and $\mathbf{b} \in F_1$. We have also shown that if $f \in G_1$ is defined by $\mathbf{b} := \{b_i\} \neq 0$, then $f(S_2) \neq S_2$, and $f(S_2) \cap S_1 = \varnothing$. If $f(S_2) \cap S_2 = \varnothing$, then S_1, $f(S_2)$, S_2 would be three pairwise disjoint sections, which contradicts to $X \not\sim Y \times \mathbb{P}^1$.

Thus S_1, $f(S_2)$, S_2 is a good configuration.

In the opposite direction: consider a good configuration S_1, S_2, S_3 such that $S_3 \cap S_1 = \varnothing$ and $S_3 \cap S_2 \neq \varnothing$. Since S_3 is a section of p and does not meet S_1, it is defined by a section $\mathbf{b} := \{b_i\}$ as $z_i = b_i(y)$, $y \in U_i$. Thus, $F_1 \neq \{0\}$.

The case of G_2 and sections that meet S_1 but do not meet S_2 can be treated in the same way, by interchanging S_2 with S_1 and F_1 with F_2. This proves 1)–3).

Let us prove 4). If for each $f \in G$ all points in $S_1 \cup S_2$ are fixed, then $G = G_0 \cong \mathbb{C}^*$ by Lemma 11.10. If this is not the case, take $\phi \in G \setminus G_0$. Then $\phi(S_1) = S_2$ and $\phi(S_2) = S_1$. Thus, $\phi(y, z_i) = a_i(y)/z_i$ in every V_i and

$$\lambda_{ij} \frac{a_i(y)}{z_i} = \frac{a_j(y)}{\lambda_{ij} z_i} \tag{50}$$

where the $a_i(y)$ are non-vanishing holomorphic functions in U_i. Thus $\{a_i(y)\}$ define a section of $\mathcal{L}^{\otimes 2}$. Since $a_i(y)$ never vanish, we get that $\mathcal{L}^{\otimes 2}$ is trivial. Therefore, we may choose z_i in such a way that $a_i = a \in \mathbb{C}^*$. Then $\phi = \phi_a$.

For any other $f \in G \setminus G_0$ the composition $f \circ \phi$ belongs to G_0, hence $G = G_0 \sqcup \phi_a \cdot G_0$. Proposition 11.19 is proved.

Corollary 11.20. *Let (X, p, Y) be a $\mathbb{P}^1$-bundle, where X and Y are compact connected manifolds and $X \not\sim Y \times \mathbb{P}^1$. Assume that p admits no good configurations but admits two disjoint sections S_1 and S_2. Then one of the following holds*:

1) $\mathrm{Aut}(X)_p \cong \mathbb{C}^*$;
2) *the holomorphic line bundle $\mathcal{L}(S_1, S_2)^{\otimes 2}$ is trivial and* $\mathrm{Aut}(X)_p = G_0 \sqcup \phi_a \cdot G_0$, *for some $a \in \mathbb{C}^*$; here $G_0 \cong \mathbb{C}^*$ and $a \in \mathbb{C}^*$.*

The restriction map $\mathrm{Aut}(X)_p \to \mathrm{Aut}(P_y)$, $f \mapsto f|_{P_y}$, *is a group embedding.*

Proof. It follows from Proposition 11.19 that $F_1 = F_2 = \{0\}$, thus $\mathrm{Aut}(X)_p = G$.

11.3. Automorphisms of type C. Let (X, p, Y) be a $\mathbb{P}^1$-bundle where X and Y are complex compact connected manifolds. Assume that $X \not\sim Y \times \mathbb{P}^1$ and $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$, has type **C**. The analytic subset $F \subset X$ of all fixed points of f contains no sections, but contains a bisection S that is a smooth unramified double cover of Y (see Lemma 11.10). Below we use the notation of Lemmas 11.10 and 11.11.

LEMMA 11.21. *Assume that $\widetilde{X} := \widetilde{X}_S \not\sim S \times \mathbb{P}^1$. Let $N \subset \widetilde{X}$ be a section of $\tilde{p}$ distinct from S_+ and S_-. Then $N_X := p_X(N)$ is a section of p and S_+, S_-, N is not a good configuration.*

PROOF. Let us show that $p_X\colon N \to N_X$ is an unramified double cover. Indeed, assume that this is not the case. Since $\widetilde{X}$ is the unramified double cover of X, the preimage $p_X^{-1}(x)$ contains precisely two points for every $x \in N_X$. Thus, if $p_X^{-1}(N_X) \neq N$, then the preimage $p_X^{-1}(N_X)$ consists of two irreducible components, N and N_1. Moreover, since p_X is unramified, we have $N \cap N_1 = \varnothing$. It follows that there are two distinct pairs of non-intersecting sections of $\tilde{p}$, namely, S_+, S_- and N, N_1. According to Lemma 11.6, $\widetilde{X} \sim S \times \mathbb{P}^1$, which gives us a contradiction. It follows that N is a double cover of N_X. Let $s \in S$ and $y = p(s) = p(\mathrm{inv}(s))$. Then

$$p_X^{-1}(N_X \cap P_y) = N \cap p_X^{-1}(P_y) = N \cap \big(\tilde{p}^{-1}(s) \cup \tilde{p}^{-1}(\mathrm{inv}(s))\big)$$

contains two points (since N meets every fibre of $\tilde{p}$ at a single point).

Since N is double cover of N_X, it follows that $(N_X \cap P_y)$ contains precisely one point. Therefore, N_X is a section of p.

Assume that N meets S_+ at a point $a = (s, s) \in \widetilde{X}$, $s \in S$. Then it meets S_- at the point $\mathrm{inv}(a) = (\mathrm{inv}(s), s)$ since $p_X(a) = p_X(\mathrm{inv}(a))$. Thus, N meets both S_+ and S_- and the configuration is not good.

COROLLARY 11.22. *Assume that (X, p, Y) is a $\mathbb{P}^1$-bundle that admits a non-identity automorphism $f \in \mathrm{Aut}(X)_p$ of type* ***C*** *with data S. Assume that the corresponding double cover $\widetilde{X}_S \not\sim S \times \mathbb{P}^1$. Then*

(i) *one of the following holds*:
- $\mathrm{Aut}(\widetilde{X})_{\tilde{p}} \cong \mathbb{C}^*$,
- $\mathrm{Aut}(\widetilde{X})_{\tilde{p}} = \widetilde{G}_0 \sqcup \phi_a \cdot \widetilde{G}_0$, *where* $\widetilde{G}_0 \cong \mathbb{C}^*$ *and* $\phi \in \mathrm{Aut}(\widetilde{X})_{\tilde{p}}$ *interchanges* S_+ *with* S_-;

(ii) *the restriction map* $\mathrm{Aut}(X)_p \to \mathrm{Aut}(P_y)$, $f \mapsto f|_{P_y}$, *is a group embedding for every* $y \in Y$;

(iii) *the map* $h \mapsto \tilde{h}$ *is a group embedding of* $\mathrm{Aut}(X)_p$ *to* $\mathrm{Aut}(\widetilde{X})_{\tilde{p}}$.

PROOF. Since there are no good configurations in $\widetilde{X}_f$ by Lemma 11.21, assertion (i) follows from Corollary 11.20 applied to $\widetilde{X}$.

Take $u \in S$ and $t \in Y$, $t = p(u)$. If $f|_{P_t} = \mathrm{id}$, then by construction

(a) $\tilde{f}\big|_{P_u} = \mathrm{id}$, hence

(b) $\tilde{f} = \mathrm{id}$ (by Corollary 11.20 applied to $\widetilde{X}$), hence

(c) $\tilde{f}\big|_{P_s} = \mathrm{id}$ for every $s \in S$, hence

(d) $f|_{P_y} = \mathrm{id}$ for $y = p(s) \in Y$.

Consequently, f is uniquely determined by its restriction to the fibre $P_t = p^{-1}(t)$. This proves (ii).

On the other hand it was shown in (ii) that $\tilde{h} = \mathrm{id}$ implies that $f|_{P_y} = \mathrm{id}$ for every $y \in Y$, that is, $h = \mathrm{id}$. Therefore, $h \mapsto \tilde{h}$ is an embedding. This proves (iii).

LEMMA 11.23. *Assume that* $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$, *and* f *is of type* **C** *with data (bisection)* S.

(i) *If the corresponding double cover (see case* **C***)* $\widetilde{X} := \widetilde{X}_S$ *is not isomorphic to* $S \times \mathbb{P}^1$, *then the group* $\mathrm{Aut}(X)_p$ *has exponent* 2 *and consists of two or four elements.*

(ii) *If* $\widetilde{X}$ *is isomorphic to* $S \times \mathbb{P}^1$, *then there are two disjoint sections* $S_1, S_2 \subset X$ *of* p. *Moreover, if* $X \not\sim Y \times \mathbb{P}^1$, *then* $\mathrm{Aut}(X)_p$ *is a disjoint union of its abelian complex Lie subgroup* $\Gamma \cong \mathbb{C}^*$ *of index* 2 *and its coset* Γ'. *The subgroup* Γ *consists of those* $f \in \mathrm{Aut}(X)_p$ *that fix* S_1 *and* S_2. *The coset* Γ' *consists of those* $f \in \mathrm{Aut}(X)_p$ *that interchange* S_1 *and* S_2. *Moreover, the restriction homomorphism* $\mathrm{Aut}(X)_p \to \mathrm{Aut}(P_y)$, $f \mapsto f|_{P_y}$, *is a group embedding for every* $y \in Y$.

PROOF. We modify the proof of Lemma 4.7 in [7].

Choose a point $a \in S$. Let $b = p(a) \in Y$. This means that a sits in the two-element set $S \cap P_b$. The lift $\tilde{f}$ of f to $\widetilde{X}$ has type **A** with data $(S_+, S_-) \subset \widetilde{X}$, since the points of S are fixed by f. It is determined uniquely by its restriction to P_a (see Proposition 11.19). For the corresponding holomorphic line bundle $\widetilde{\mathcal{L}} := \widetilde{\mathcal{L}}(S_-, S_+)$ the section S_+ is the zero section. Let

- $\{\widetilde{U}_j\}$ be a fine covering of S;
- (u, z_j) be local coordinates in $\widetilde{V}_j = \tilde{p}^{-1}(\widetilde{U}_j)$ such that $z_j\big|_{S_+} = 0$, $z_j\big|_{S_-} = \infty$;
- $a \in \widetilde{U}_i$, $\mathrm{inv}(a) \in \widetilde{U}_k$, and $\widetilde{U}_k \cap \widetilde{U}_i = \varnothing$;
- $b = p(a) = p(\mathrm{inv}(a)) \in Y$.

The following two assertions were proved in Lemma 4.7 of [7].

A. If we define the isomorphism $\alpha\colon \overline{\mathbb{C}}_{z_i} \to \overline{\mathbb{C}}_{z_k}$ in such a way that the diagram

$$\begin{tikzcd} P_b \arrow[r, "(a,\mathrm{id})"] \arrow[d, "\mathrm{id}"] & a \times P_b \arrow[r, "z_i"] \arrow[d] & \overline{\mathbb{C}}_{z_i} \arrow[d, "\alpha"] \\ P_b \arrow[r, "(\mathrm{inv}(a),\mathrm{id})"] & \mathrm{inv}(a) \times P_b \arrow[r, "z_k"] & \overline{\mathbb{C}}_{z_k} \end{tikzcd} \tag{51}$$

is commutative, then

$$z_k = \alpha(z_i) = \frac{\nu}{z_i}$$

for some $\nu = \nu(a) \neq 0$.

B. Consider an automorphism $h \in \mathrm{Aut}(X)_p$. Let $\tilde{h}$ be its pullback to $\mathrm{Aut}(\widetilde{X})_{\tilde{p}}$ defined by $\tilde{h}(s, x) = (s, h(x))$. Let $n_1(z_i) = \tilde{h}\big|_{\widetilde{P}_a}$, which means that $h(a, z_i) = (a, n_1(z_i))$. Let $n_2(z_k) = \tilde{h}\big|_{\widetilde{P}_{\mathrm{inv}(a)}}$, which means that $h(\mathrm{inv}(a), z_k) = (a, n_2(z_k))$. Then

$$\frac{\nu}{n_1(z_i)} = \alpha(n_1(z_i)) = n_2(\alpha(z_i)) = n_2\left(\frac{\nu}{z_i}\right). \tag{52}$$

Proof of (i). Assume that $\widetilde{X} \not\sim S \times \mathbb{P}^1$.

According to Corollary 11.22, if $\tilde{h} \in \mathrm{Aut}(\widetilde{X})_{\tilde{p}}$, then either $\tilde{h}(s, z_j) = \lambda z_j$, or $h(s, z_j) = \lambda/z_j$ in every $\widetilde{U}_j$ of our fine covering, where $\lambda \in \mathbb{C}^*$ does not depend on s or j.

Fix $a \in S$. According to item B one of following two conditions holds:

(a) $n_1(z_i) = \lambda z_i$, $n_2(z_k) = \lambda z_k$, $z_k = \nu(a)/z_i$, and

$$\frac{\nu(a)}{\lambda z_i} = \lambda \frac{\nu(a)}{z_i}$$

by (52);

(b) $n_1(z_i) = \lambda/z_i$, $n_2(z_k) = \lambda/z_k$, $z_k = \nu/z_i$, and

$$\frac{\nu z_i}{\lambda} = \frac{\lambda z_i}{\nu}$$

by (52).

In the former case $\lambda = \pm 1$ and in the latter $\lambda = \pm\nu$. Hence at most four maps are possible. Clearly, the squares of all these maps are the identity map.

Note that all calculations are done for the fibre of $\tilde{p}$ over the point a. We use the fact that the map $\tilde{h}$ is defined by its restriction to a fibre. *A priori*, ν could depend on the fibre. But since λ does not, as a byproduct we obtain that the same is valid for ν.

Proof of (ii). Assume that $\widetilde{X} \sim S \times \mathbb{P}^1$. Let $\zeta\colon S \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection onto the second factor, and let $\zeta_1 = \zeta|_{S_+}$ and $\zeta_2 = \zeta|_{S_-}$. Since $S_+ \cap S_- = \varnothing$, the function $z = (\zeta - \zeta_1)/(\zeta - \zeta_2)$ is well defined on $\widetilde{X}$.

Since $z = 0$ on $S_+ = \{(s, s)\}$ and $z = \infty$ on $S_- = \{(s, \mathrm{inv}(s))\}$, we may assume that $z_j = z$ for all j. Recall that for every s

$$\mathrm{inv}(s, z) = (\mathrm{inv}(s), \alpha(z)) = \left(\mathrm{inv}(s), \frac{\nu(s)}{z}\right). \tag{53}$$

This implies that $\nu(s)$ is a holomorphic function on S, hence $\nu = \mathrm{const}$. From (53) we get that two disjoint sections $N_1 = \{(s,\, z = \sqrt{\nu})\}$ and $N_2 = \{(s,\, z = -\sqrt{\nu})\}$ (for some choice of $\sqrt{\nu}$) are invariant under the involution, which means that their images are two disjoint sections S_1 and S_2, respectively, in X.

Thus, X has two disjoint sections. Let us show that there is no good configuration in X. Assume that S_3 is a third section (of p) in X. On $\widetilde{S}_3 = p_X^{-1}(S_3) \subset \widetilde{X}$ the function z is either a constant or takes all values in $\overline{\mathbb{C}}$. If it is constant, then X has three disjoint sections (S_1, S_2, S_3), thus $X = Y \times \mathbb{P}^1$. If z takes all values on $\widetilde{S}_3$, then S_3 meets both S_1 and S_2, thus S_1, S_2, S_3 is not a good configuration.

Now (ii) follows from Corollary 11.20, which completes the proof of Lemma 11.23.

We have proved (see Lemma 11.12) that if $X \not\sim Y \times \mathbb{P}^1$ and there is $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$, of type **B**, then $\mathrm{Aut}(X)_p$ contains a subgroup isomorphic to $(\mathbb{C}^+)^n$ for some positive integer n.

Corollary 11.24. *Assume that $X \not\sim Y \times \mathbb{P}^1$ and $\mathrm{Aut}(X)_p$ contains an automorphism $f \neq \mathrm{id}$ of type* **B**. *Then $\mathrm{Aut}(X)_p$ contains no automorphisms of type* **C**.

Proof. Assume that $\operatorname{Aut}(X)_p$ contains an automorphism of type **C**. Then by Lemma 11.23, $\operatorname{Aut}(X)_p$ is either finite or consists of two cosets isomorphic to $\mathbb{C}^*$; in both cases $\operatorname{Aut}(X)_p$ does not contain a Lie subgroup $\Gamma \cong (\mathbb{C}^+)^n$ with $n > 0$.

Proposition 11.25. *Let (X, p, Y) be a $\mathbb{P}^1$-bundle, where X and Y are complex compact connected manifolds and Y is Kähler and not uniruled. Then* $\operatorname{Aut}(X)$ *is Jordan.*

Proof. Indeed, we have proved that three cases are possible.

(a) $\operatorname{Aut}(X)_p = \{\mathrm{id}\}$; then $\operatorname{Aut}(X)$ embeds in $\operatorname{Aut}(Y)$, which is Jordan according to [34].

(b) $\operatorname{Aut}(X)_p$ contains an automorphisms of type **A** or **B**. Then $X = \mathbb{P}(\mathcal{E})$ for some rank 2 vector bundle $\mathcal{E}$ on Y. Thus, X is Kähler [82; Proposition 3.5].

(c) $\operatorname{Aut}(X)_p$ contains an automorphisms of type **C**. Then the double cover $\widetilde{X}$ of X fits into case (b). Thus, X is Kähler.

In cases (b) and (c), $\operatorname{Aut}(X)$ is Jordan, once more, according to [34].

12. Structure of $\operatorname{Aut}_0(X)$ and $\operatorname{Aut}(X)$

In this section we prove the main result of this chapter. Namely, we prove that the group $\operatorname{Aut}(X)$ is very Jordan, provided that the $\mathbb{P}^1$-bundle (X, p, Y) is scarce.

Theorem 12.1. *Let (X, p, Y) be a $\mathbb{P}^1$-bundle, where X, Y are complex compact connected manifolds, X is not biholomorphic to the direct product $Y \times \mathbb{P}^1$, and Y is Kähler and not uniruled. Assume that (X, p, Y) is scarce.*

Then:

(i) *the connected identity component* $\operatorname{Aut}_0(X)$ *of the complex Lie group* $\operatorname{Aut}(X)$ *is* commutative;

(ii) *the group* $\operatorname{Aut}(X)$ *is very Jordan*; *more precisely*, *there is a short exact sequence*

$$1 \to \operatorname{Aut}_0(X) \to \operatorname{Aut}(X) \to F \to 1, \tag{54}$$

where F is a bounded group;

(iii) *the commutative group* $\operatorname{Aut}_0(X)$ *sits in a short exact sequence of complex Lie groups*

$$1 \to \Gamma \to \operatorname{Aut}_0(X) \to H \to 1, \tag{55}$$

where H is a complex torus and one of the following conditions holds:

(a) $\Gamma = \{\mathrm{id}\}$, *the trivial group*;

(b) $\Gamma \cong (\mathbb{C}^+)^n$;

(c) $\Gamma \cong \mathbb{C}^*$.

Proof. We know that the set of almost sections is either infinite or contains at most two of them (by Lemma 11.8 and Remark 6.6).

Consider two cases.

Case 1. There are no almost sections of p. Then $\operatorname{Aut}(X)_p$ is finite by Lemma 11.23.

Case 2. p has only two almost sections, A_1 and A_2, which meet.

Assume that $f \in \operatorname{Aut}(X)_p$, $f \neq \mathrm{id}$. Since f takes almost sections to almost sections, $A_1 \cup A_2$ is invariant under f. According to Proposition 11.19, the following cases are possible:

- Points of A_1 are fixed points of f. Then the same is true for A_2. Since A_1 and A_2 meet, f is neither of type **A** nor of type **C**. Since they are distinct, f cannot be of type **B** (see Lemma 11.16). Thus $f = \mathrm{id}$ and $\operatorname{Aut}(X)_p = \{\mathrm{id}\}$.
- Not all points of A_1 are fixed points of f. This means that $f(A_1) = A_2$ and $f(A_2) = A_1$. Assume that $g \neq f \in \operatorname{Aut}(X)_p$ and $g \neq \mathrm{id}$. Since $g \neq \mathrm{id}$, it does not fix points of A_1 either (by the previous case). Then for $h := g \circ f$ we have $h(A_1) = A_1$, $h(A_2) = A_2$. Hence, as in the previous paragraph, $h = \mathrm{id}$. It follows that $f^2 = \mathrm{id}$, $g = f = f^{-1}$.

Thus in this case $\operatorname{Aut}(X)_p$ is finite.

Case 3: p has precisely one almost section. Then there are no automorphisms of type **A**, since there are no two disjoint sections. If $\operatorname{Aut}(X)_p$ contains no automorphisms of type **B**, then $\operatorname{Aut}(X)_p$ is finite by Lemma 11.23. If $\operatorname{Aut}(X)_p$ contains an automorphism of type **B**, then, thanks to Corollary 11.24, $\operatorname{Aut}(X)_p$ contains no automorphisms of type **C**. Since all automorphisms of type **B** have to share this section in their sets of fixed points, $\operatorname{Aut}(X)_p \cong (\mathbb{C}^+)^n$ by Proposition 11.13 (unless $\operatorname{Aut}(X)_p = \{\mathrm{id}\}$).

Case 4: p admits precisely two almost sections S_1 and S_2 and they do not meet. Then they are sections. But X admits no good configuration. Thus, by Proposition 11.19 the group $\operatorname{Aut}(X)_p$ contains a subgroup isomorphic to $\mathbb{C}^*$ of index at most 2.

Case 5: X is scarce and all almost sections meet pairwise (in particular, all sections meet pairwise). Then $\operatorname{Aut}(X)_p$ contains no automorphism of type **A**. If $\operatorname{Aut}(X)_p$ contains an automorphism of type **B**, then, by Lemma 11.16, the set of sections cannot be scarce (provided that there is more than one of them), which is contradiction. Hence $\operatorname{Aut}(X)_p$ is finite by Lemma 11.23.

Case 6: X is scarce and admits two disjoint sections S_1 and S_2. By Lemma 11.9, X admits no good configurations, and by Lemma 11.16 it has no automorphisms of type **B**. By Corollary 11.20, $\operatorname{Aut}(X)_p$ contains a subgroup isomorphic to $\mathbb{C}^*$ of index at most 2.

The proof now repeats the proof of Theorem 5.4 in [7] with only one modification: $\mathbb{C}^+$ should be changed to $(\mathbb{C}^+)^n$ and, accordingly, Lemma 2.10 should be applied. The group $\operatorname{Aut}(X)_p$ can be included in the short exact sequence

$$1 \to (\operatorname{Aut}(X)_p \cap \operatorname{Aut}_0(X)) \to \operatorname{Aut}_0(X) \xrightarrow{\tau} H_0 \to 1, \tag{56}$$

where $H_0 = \tau(\operatorname{Aut}_0(X)) \subset \operatorname{Tor}(Y)$ is a torus (see Remark 5.6). According to Cases 1–6, one of the following holds:

- $\operatorname{Aut}(X)_p \cap \operatorname{Aut}_0(X)$ is finite (thus $\operatorname{Aut}_0(X)$ is a complex torus);
- $\operatorname{Aut}(X)_p \cap \operatorname{Aut}_0(X) \cong (\mathbb{C}^+)^n$;
- $\operatorname{Aut}(X)_p \cap \operatorname{Aut}_0(X) \cong (\mathbb{C}^*)$.

Thus, by Lemma 2.10 the group $\operatorname{Aut}_0(X)$ is commutative. Now the theorem follows from the fact that $\operatorname{Aut}(X)/\operatorname{Aut}_0(X)$ is bounded (see Proposition 3.5).

13. Rational bundles over poor manifolds

In this section we consider rational bundles over poor manifolds. We prove that if Y is poor, then p is scarce and the results of the previous section apply.

DEFINITION 13.1. We say that a compact connected complex manifold Y of positive dimension is *poor* if it enjoys the following properties:

- Y does not contain analytic subspaces of codimension 1 (*a fortiori*, the algebraic dimension $a(Y)$ of Y is 0);
- Y does not contain rational curves, that is, it is meromorphically hyperbolic in the sense of Fujiki [24].

A complex torus T with $\dim(T) \geqslant 2$ and $a(T) = 0$ is a poor Kähler manifold. Indeed, a complex torus T is a Kähler manifold that does not contain rational curves. If $a(T) = 0$, it contains no analytic subsets of codimension 1 [11; Chap. 2, Corollary 6.4]. An explicit example of such a torus of dimension 2 is given in [11; Example 7.4]. Explicit examples of poor tori of any dimension are presented in [8]. Another example of a poor manifold is provided by a non-algebraic $K3$ surface S with the Néron–Severi group $\mathrm{NS}(S) = 0$ (see [9; Chap. VIII, Proposition 3.6]).

Below Y is assumed to be a compact connected complex manifold.

PROPOSITION 13.2 [7; Proposition 3.6]. *Let (X, p, Y) be an equidimensional rational bundle. Assume that Y contains no analytic subsets of codimension* 1. *Then (X, p, Y) is a $\mathbb{P}^1$-bundle.*

PROOF. Let $\dim(Y) = n$, and let

$$S = \{x \in X \mid \mathrm{rk}(dp)(x) < n\}$$

be the set of all points in X where the differential dp of p does not have the maximum rank. Then S and $\widetilde{S} = p(S)$ are analytic subsets of X and Y, respectively (see, for instance, [54; Chap. VII, Theorem 2], [57; Theorem 1.22], [73]). Moreover, $\mathrm{codim}(\widetilde{S}) = 1$ (see [72]). Since Y contains no analytic subsets of codimension 1, we obtain: $\widetilde{S} = \varnothing$. Thus the holomorphic map p has no singular fibres.

LEMMA 13.3. *Let (X, p, Y) be a $\mathbb{P}^1$-bundle such that* $\dim(Y) = n$. *For an almost section A set $\Sigma(A) = \{y \in Y \mid p^{-1}(y) \subset A\}$. If Y contains no analytic subsets of codimension* 1, *then*

(i) *any n-section has no ramification points* (*i.e.*, *the intersection $X \cap P_y$ consists of n distinct points for every $y \in Y$*);

(ii) *if A_1 and A_2 are two almost sections*, *then* $p(A_1 \cap A_2) \subset \Sigma(A_1) \cap \Sigma(A_2)$;

(iii) *any two distinct sections of p in X are disjoint*;

(iv) *if there is an almost section $A \subset X$ that is not a section*, *then X contains neither sections nor n-sections.*

PROOF. (i) Let R be an n-section of p, and let A be the set of all points $x \in R$ where the restriction $p|_R \colon R \to Y$ of p to R is not locally biholomorphic. Then the image $p(A)$ is either empty or has pure codimension 1 in Y ([20; § 1, 9], [56; Theorem 1.6], [73]). Since Y contains no analytic subsets of codimension 1, we have $p(A) = \varnothing$. Hence, $A = \varnothing$.

(ii) Let B be an irreducible component of $A_1 \cap A_2$. Since $\dim(B) = n-1$ and $\dim(p(B)) \leqslant n-2$, we have $p^{-1}(p(b)) \subset B$ for every point $b \in B$. Thus, $p(b) \in \Sigma(A_1) \cap \Sigma(A_2)$.

(iii) In particular, if A_1 and A_2 are distinct sections, then $\Sigma(A_1) = \Sigma(A_2) = \varnothing$ and $A := A_1 \cap A_2 = \varnothing$.

(iv) Since A is not a section, there is a point $y \in Y$ such that $P_y = p^{-1}(y) \subset A$. Thus for any n-section S we have $S \cap A \neq \varnothing$. This contradicts assertion (ii) since $\Sigma(S) = \varnothing$. Hence such a section S does not exist.

COROLLARY 13.4. *Let* (X, p, Y) *be a* $\mathbb{P}^1$*-bundle*, $\dim(Y) = n$. *If* Y *contains no analytic subsets of codimension* 1, *then one of the following holds*:

(i) $X \sim Y \times \mathbb{P}^1$;

(ii) X *admits two disjoint sections*, *and* $\operatorname{Aut}(X)_p$ *contains a subgroup* $G \cong \mathbb{C}^*$ *of index at most* 2;

(iii) X *admits two meeting almost sections*, *and* $\operatorname{Aut}(X)_p$ *is finite*;

(iv) X *admits precisely one almost section* D; *then either* $\operatorname{Aut}(X)_p \cong \mathbb{C}^+$ (*and* D *is a section by Lemma* 11.14) *or* $\operatorname{Aut}(X)_p = \{\mathrm{id}\}$;

(v) X *admits no almost sections and* $\operatorname{Aut}(X)_p$ *is finite.*

PROOF. First, note that since Y does not admit meromorphic functions, given a line bundle $\mathcal{L}$ on Y, either $H^0(\mathcal{L}) = \{0\}$ or $\mathcal{L}$ is trivial and $H^0(\mathcal{L}) \cong \mathbb{C}$.

Item (i): Assume that X admits $m \geqslant 3$ almost sections. By Lemma 13.3 they are disjoint over an open set $U \subset Y$ that has a complement of codimension 2. Thus, $X \sim Y \times \mathbb{P}^1$ by Lemma 11.2.

Item (ii) follows from Corollary 11.20.

Item (iii) is proved in Case 3 of the proof of Theorem 12.1.

Item (iv) follows from Proposition 11.13: if $\operatorname{Aut}(X)_p \neq \{\mathrm{id}\}$, then $\operatorname{Aut}(X)_p$ is isomorphic to the additive group of $\mathbb{C}^m$. This means that for corresponding line bundle we have $0 < m = H^0(\mathcal{L})$. Hence $m = 1$.

Item (v) follows from Lemma 11.23.

LEMMA 13.5. *Let* (X, p, Y) *be a* $\mathbb{P}^1$*-bundle such that* $\dim(Y) = n$. *If* Y *is poor*, *then* $\operatorname{Bim}(X) = \operatorname{Aut}(X)$.

PROOF. Since Y contains no rational curves, it is not uniruled. According to Corollary 5.5, every map $f \in \operatorname{Bim}(X)$ is p-fibrewise, that is, there exists a group homomorphism $\tilde{\tau}\colon \operatorname{Bim}(X) \to \operatorname{Bim}(Y)$ (see Lemma 5.4) such that for all $f \in \operatorname{Bim}(X)$

$$p \circ f = \tilde{\tau}(f) \circ p.$$

Since Y contains no rational curves, every meromophic map into Y is holomorphic ([24], also see Remark 3.4). Thus $\tilde{\tau}(f) \in \operatorname{Aut}(Y)$.

For $f \in \operatorname{Bim}(X)$ let $\widetilde{S}_f$ be the indeterminacy locus of f, which is an analytic subspace of X of codimension at least 2 [73; p. 369]. Let $S_f = p(\widetilde{S}_f)$, which is an analytic subset of Y (see [73], [54; Chap. VII, Theorem 2]). Since Y contains no analytic subsets of codimension 1, $\operatorname{codim}(S_f) \geqslant 2$. Moreover, f is defined at all points of $X \setminus p^{-1}(S_f)$. By Lemma 5.12 both $f \in \operatorname{Bim}(X)$ and $f^{-1} \in \operatorname{Bim}(X)$ can be extended to X holomorphically, hence $\operatorname{Bim}(X) = \operatorname{Aut}(X)$.

We summarize the result in the following theorem.

THEOREM 13.6. *Let (X, p, Y) be an equidimensional rational bundle over a poor Kähler manifold Y. Then*

- (X, p, Y) *is a* $\mathbb{P}^1$*-bundle* (*see Proposition* 13.2);
- $\mathrm{Bim}(X) = \mathrm{Aut}(X)$ (*see Lemma* 13.5).

Assume additionally that Y is Kähler and X is not isomorphic to the direct product $Y \times \mathbb{P}^1$. Then

- *X admits at most two almost sections* (*Corollary* 13.4);
- *the connected identity component* $\mathrm{Aut}_0(X)$ *of the complex Lie group* $\mathrm{Aut}(X)$ *is* commutative (*Theorem* 12.1);
- *the group* $\mathrm{Aut}(X)$ *is very Jordan* (*Theorem* 12.1);
- *the commutative group* $\mathrm{Aut}_0(X)$ *sits in a short exact sequence of complex Lie groups*
$$1 \to \Gamma \to \mathrm{Aut}_0(X) \to H \to 1, \tag{57}$$
where H is a complex torus and one of the following conditions holds (*Corollary* 13.4):
  (a) $\Gamma = \{\mathrm{id}\}$, *the trivial group*;
  (b) $\Gamma \cong \mathbb{C}^+$, *the additive group of complex numbers*;
  (c) $\Gamma \cong \mathbb{C}^*$, *the multiplicative group of complex numbers*.

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Tatiana Bandman
(Татьяна Бандман)
Bar-Ilan University, Department of Mathematics,
Ramat Gan, Israel
E-mail: bandman@math.biu.ac.il

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Yuri G. Zarhin
(Юрий Геннадьевич Зархин)
Pennsylvania State University, Department of
Mathematics, University Park, PA, USA
E-mail: zarhin@math.psu.edu